DISCRETE SERIES CHARACTERS FOR AFFINE HECKE ALGEBRAS AND THEIR FORMAL DEGREES

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Abstract. We introduce the *generic central character* of an irreducible discrete series representation of an affine Hecke algebra. Using this invariant we give a new classification of the irreducible discrete series characters for all abstract affine Hecke algebras (except for the types $E^{(1)}_{6,7,8}$) with *arbitrary positive parameters* and we prove an explicit product formula for their formal degrees (in all cases).

Contents

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1. Introduction

Considering the role of affine Hecke algebras in representation theory [IM], [Bo], [BZ], [BM1], [BM2], [Mo1], [Mo2], [L2], [R1], [BKH], [BK] or in the theory of integrable models [C], [HO1], [Ma2], [EOS] it is natural to ask for the description of their (algebraic) representation theory and for the properties of their representations in relation to harmonic analysis (e.g. unitarity, temperedness, formal degrees). An analytic approach to such questions (based on the spectral theory of C^*-algebras) was first proposed by Matsumoto [Mat]. This approach to affine Hecke algebras gives rise to a program in the spirit of Harish-Chandra's work on the harmonic analysis on locally compact groups arising from reductive groups (for a concise account of Harish-Chandra's work in the p-adic case see [W]). The main challenges to surmount on this classical route designed to describe the tempered spectrum and the Plancherel isomorphism (the "philosophy of cups forms") are related to understanding the basic building blocks, the so-called discrete series characters. The most fundamental problems are:

(i) Classify the irreducible discrete series characters.
(ii) Calculate their formal degrees.

In the present paper we will essentially[1] solve both these problems for general abstract semisimple affine Hecke algebras with *arbitrary* positive parameters.

The study of harmonic analysis in this context requires the introduction of classical notions borrowed from Harish-Chandra's seminal work (e.g. the Schwartz completion, temperedness, parabolic induction) for abstract affine Hecke algebras. It was shown in [DO] that the above program can indeed be carried out. In view of [DO] (also see [O2]) our solution of (i) can in fact be amplified to yield the classification of all irreducible tempered characters of the Hecke algebra. The explicit Plancherel isomorphism can be reconstructed by (ii) and [O1, Theorem 4.43].

Let us describe the methods used in this paper. The new tool in this study of these questions for abstract affine Hecke algebras is derived from the presence of a space of continuous parameters with respect to which the harmonic analysis naturally deforms. Observe that this aspect is missing in the traditional context of the harmonic analysis on reductive groups. The main message of this paper is that parameter deformation is a powerful tool for solving the questions (i) and (ii), especially (but not exclusively) for non-simply laced root data. There are in fact two other pillars on which our method rests, based on results from [O1] and [OS]. We will now give a more detailed account of these matters.

An affine Hecke algebra $\mathcal{H} = \mathcal{H}(\mathcal{R}, q)$ is defined in terms of a based root datum

$$\mathcal{R} = (X, R_0, Y, R_0^\vee, F_0)$$

and a parameter function $q \in \mathcal{Q} = \mathcal{Q}(\mathcal{R})$. By this we mean that q is a (positive) function on the set S of simple affine reflections in the affine Weyl group $\mathbb{Z}R_0 \rtimes W_0$, such that $q(s) = q(s')$ whenever s and s' are conjugate in the extended Weyl group $W = X \rtimes W_0$. The deformation method is based on regarding the affine Hecke algebras $\mathcal{H}(\mathcal{R}, q)$ with fixed $\mathcal{R}$ as a continuous field of algebras, depending on the

[1] Our solution of (i) does not cover the cases E_n ($n = 6, 7, 8$), hence in these cases we rely on [KL]. Our solution of (ii) is complete only up to the determination of a rational constant factor for each continuous family (in the sense to be explained below) of discrete series characters.

parameter q. This enables us to transfer properties that hold for $q \equiv 1$ or for *generic* q to *arbitrary* positive parameters.

We will prove that every irreducible discrete series character δ_0 of $\mathcal{H}(\mathcal{R}, q_0)$ is the evaluation at q_0 of a *unique* maximal continuous family $q \to \delta_q$ of discrete series characters of $\mathcal{H}(\mathcal{R}, q)$ defined in a suitable open neighborhood of q_0. The continuity of the family means that the corresponding family of primitive central idempotents $q \to e_\delta(q) \in \mathcal{S}$ (the Schwartz completion of $\mathcal{H}(\mathcal{R}, q)$, a Fréchet algebra which is independent of q as a Fréchet space) is continuous in q with respect to the Fréchet topology of $\mathcal{S}$. The maximal domain of definition of the family $q \to \delta_q$ is described in terms of the zero locus of an explicit rational function on $\mathcal{Q}$. This reduces the classification of the discrete series of $\mathcal{H}(\mathcal{R}, q)$ for *arbitrary* (possibly special) positive parameters to that for *generic* positive parameters, a problem that is considerably easier than the general case.

Let us take the discussion one step further to see how this idea leads to a practical strategy for the classification of the discrete series characters. For this it is crucial to understand how the "central characters" behave under the unique continuous deformation $q \to \delta_q$ of an irreducible discrete series character δ_0. Since it is known that the set of discrete series can be nonempty only if R_0 spans $X \otimes_{\mathbb{Z}} \mathbb{Q}$, we assume this throughout the paper. To enable the use of analytic techniques we need an involution * and a positive trace τ on our affine Hecke algebras $\mathcal{H}(\mathcal{R}, q)$. A natural choice is available, provided that all parameters are positive (another assumption we make throughout this paper). Then $\mathcal{H}(\mathcal{R}, q)$ is in fact a Hilbert algebra with tracial state τ. The spectral decomposition of τ defines a positive measure μ_{Pl} (called the Plancherel measure) on the set of irreducible representations of $\mathcal{H}(\mathcal{R}, q)$, cf. [O1, DO]. More or less by definition an irreducible representation π belongs to the discrete series if $\mu_{Pl}(\{\pi\}) > 0$. It is known that this condition is equivalent to the statement that π is an irreducible projective representation of $\mathcal{S}(\mathcal{R}, q)$, the Schwartz completion of $\mathcal{H}(\mathcal{R}, q)$. In particular π is an irreducible discrete series representation iff π is afforded by a primitive central idempotent $e_\pi \in \mathcal{S}(\mathcal{R}, q)$ of finite rank. Thus the definition of continuity of a family of irreducible characters in the preceding paragraph makes sense for discrete series characters only. We denote the finite set of irreducible discrete series characters of $\mathcal{H}(\mathcal{R}, q)$ by $\Delta(\mathcal{R}, q)$.

A cornerstone in the spectral theory of the affine Hecke algebra is formed by Bernstein's classical construction of a large commutative subalgebra $\mathcal{A} \subset \mathcal{H}(\mathcal{R}, q)$ isomorphic to the group algebra $\mathbb{C}[X]$. It follows from this construction that the center of $\mathcal{H}(\mathcal{R}, q)$ equals $\mathcal{A}^{W_0} \simeq \mathbb{C}[X]^{W_0}$. Therefore we have a central character map

$$cc : \mathrm{Irr}(\mathcal{H}(\mathcal{R}, q)) \to W_0 \backslash T \tag{1}$$

(where T is complex torus $T = \mathrm{Hom}(X, \mathbb{C}^\times)$) which is an invariant in the sense that this map is constant on equivalence classes of irreducible representations.

It was shown by "residue calculus" [O1, Lemma 3.31] that a given orbit $W_0 t \in W_0 \backslash T$ is the the central character of a discrete series representation iff $W_0 t$ is a W_0-orbit of so-called *residual points of T*. These residual points are defined in terms of the poles and zeros of an explicit rational differential form on T (see Definition 2.39), and they have been classified completely. They depend on a pair $(\mathcal{R}, q)$ consisting of a (semisimple) root datum $\mathcal{R}$ and a parameter $q \in \mathcal{Q}$. In fact, given a semisimple root datum $\mathcal{R}$ there exist finitely many $\mathcal{Q}$-valued points r of T, called *generic residual points*, such that on a Zariski-open set of the parameter space $\mathcal{Q}$

the evaluation $r(q) \in T$ is a residual point for $(\mathcal{R}, q)$. Moreover, for every $q_0 \in \mathcal{Q}(\mathcal{R})$ and every residual point r_0 of $(\mathcal{R}, q_0)$ there exists at least one generic residual point r such that $r_0 = r(q_0)$.

For fixed $q_0 \in \mathcal{Q}$ these techniques do in general not shine any further light on the cardinality of $\Delta(\mathcal{R}, q_0)$. The problem is a well known difficulty in representation theory: the central character invariant $cc(\delta_0)$ is not strong enough to separate the equivalence classes of irreducible (discrete series) representations. But this is precisely the point where the deformation method is helpful. The idea is that at generic parameters the separation of the irreducible discrete series characters by their central character is much better (almost perfect in fact, see below) than for special parameters. Therefore we can improve the quality of the central character invariant for $\delta_0 \in \Delta(\mathcal{R}, q_0)$ by considering the family of central characters $q \to cc(\delta_q)$ of the unique continuous deformation $q \to \delta_q$ of δ_0 as described above. It turns out that this family of central characters is in fact a W_0-orbit $W_0 r$ of generic residual points. We call this the *generic central character* $gcc(\delta_0) = W_0 r$ of δ_0.

Our proof of this fact requires various techniques. First of all the existence and uniqueness of the germ of continuous deformations of a discrete series character depends in an essential way on the continuous field of pre-C^*-algebras $\mathcal{S}(\mathcal{R}, q)$, where q runs through $\mathcal{Q}$ and $\mathcal{S}(\mathcal{R}, q)$ is the Schwartz completion of $\mathcal{H}(\mathcal{R}, q)$ (see [DO]). Pick $\delta_0 \in \Delta(\mathcal{R}, q_0)$ with central character $cc(\delta_0) = W_0 r_0 \in W_0 \backslash T$. With analytic techniques we prove that there exists an open neighborhood $U \times V \subset Q \times W_0 \backslash T$ of $(q_0, W_0 r_0)$ such that (see Lemma 3.2, Theorem 3.3 and Theorem 3.4):

- there exists a unique continuous family $U \ni q \to \delta_q \in \Delta(\mathcal{R}, q)$ with $\delta_{q_0} = \delta_0$,
- the cardinality of $\{\delta \in \Delta(\mathcal{R}, q) \mid cc(\delta) \in V\}$ is independent of $q \in U$.

Next we consider the formal degree $\mu_{Pl}(\{\delta_q\})$ of $\delta_q \in \Delta(\mathcal{R}, q)$. In [OS] we proved an "index formula" for the formal degree, expressing $\mu_{Pl}(\{\delta_q\})$ as alternating sum of formal degrees of characters of certain finite dimensional involutive subalgebras of $\mathcal{H}(\mathcal{R}, q)$. It follows that $\mu_{Pl}(\{\delta_q\})$ is a rational function of $q \in U$, with rational coefficients. On the other hand using the residue calculus [O1] we derive an explicit factorization

$$\mu_{Pl}(\{\delta_q\}) = d_\delta m_{W_0 r}(q) \qquad q \in U \, , \tag{2}$$

with $d_\delta \in \mathbb{Q}^\times$ independent of q and $m_{W_0 r}(q)$ depending only on q and on the central character $cc(\delta_q) = W_0 r(q)$ (for the definition of m see (40)). Using the classification of generic residual points we prove that $q \to cc(\delta_q)$ is not only continuous but in fact (in a neighborhood of q_0) of the form $q \to W_0 r(q)$ for a unique orbit of generic residual points which we call the generic central character $gcc(\delta_0) = W_0 r$ of δ_0. Thus we can now write (2) in the form (see Theorem 5.12):

$$\mu_{Pl}(\{\delta_q\}) = d_\delta m_{gcc(\delta)}(q) \qquad q \in U \, , \tag{3}$$

where $m_{gcc(\delta)}$ is an explicit rational function with rational coefficients on $\mathcal{Q}$, which is regular on $\mathcal{Q}$ and whose zero locus is a finite union of hyperplanes in $\mathcal{Q}$ (viewed as a vector space).

The incidence space $\mathcal{O}(\mathcal{R})$ consisting of pairs $(W_0 r, q)$ with $W_0 r$ an orbit of generic residual points and $q \in \mathcal{Q}$ such that $r(q)$ is a residual point for $(\mathcal{R}, q)$ can alternatively be described as $\mathcal{O}(\mathcal{R}) = \{(W_0 r, q) \mid m_{W_0 r}(q) \neq 0\}$. Thus $\mathcal{O}(\mathcal{R})$ is a disjoint union of copies of certain convex open cones in $\mathcal{Q}$. The above deformation arguments

culminate in Theorem 5.7 stating that the map

$$GCC : \coprod_{q\in\mathcal{Q}(\mathcal{R})} \Delta(\mathcal{R},q) \to \mathcal{O}(\mathcal{R}) \tag{4}$$
$$\Delta(\mathcal{R},q) \ni \delta \to (gcc(\delta),q)$$

gives $\Delta(\mathcal{R}) := \coprod_{q\in\mathcal{Q}(\mathcal{R})} \Delta(\mathcal{R},q)$ the structure of a locally constant sheaf of finite sets on $\mathcal{O}(\mathcal{R})$. Since every component of $\mathcal{O}(\mathcal{R})$ is contractible this result reduces the classification of the set $\Delta(\mathcal{R})$ to the computation of the multiplicities of the various components of $\mathcal{O}(\mathcal{R})$ (i.e. the cardinalities of the fibers of the map GCC).

One more ingredient is of great technical importance. Lusztig [L1] proved fundamental reduction theorems which reduce the classification of irreducible representations of affine Hecke algebras effectively to the the classification of irreducible representations of degenerate affine Hecke algebras (extended by a group acting through diagram automorphisms, in general). In this paper we make frequent use of a version of these results adapted to suit the situation of arbitrary positive parameters (see Theorem 2.6 and Theorem 2.8). These reductions respect the notions of temperedness and discreteness of a representation. Using this type of results it suffices to compute the multiplicities of the *positive* components of $\mathcal{O}(\mathcal{R})$ or equivalently, to compute the multiplicities of the corresponding components in the parameter space of a degenerate affine Hecke algebra (possibly extended by a group acting through of diagram automorphisms).

The results are as follows. If R_0 is simply laced the then generic central character map itself does not contain new information compared to the ordinary central character. However with a small enhancement the generic central character map gives a complete invariant for the discrete series of D_n as well, using that the degenerate affine Hecke algebra of type D_n twisted by a diagram involution is a specialization of the degenerate affine Hecke algebra of type B_n. With this enhancement understood we can state that the generic central character is a complete invariant for the irreducible discrete series characters of a degenerate affine Hecke algebra associated with a simple root system R_0 except when R_0 is of type F_4 (in which case there exist precisely two irreducible discrete series characters which have the same generic central character (unless one of the parameters is 0)) or when R_0 is of type E_n ($n = 6, 7, 8$).

Our solution to problem (i) is listed in Sections 7 and 8. This covers essentially all cases except type E_n ($n = 6, 7, 8$) (in which cases we rely on [KL] for the classification). In this classification the irreducible discrete series characters are parametrized in terms of their generic central character. The solution to problem (ii) is given by the product formula (3) (see Theorem 5.12) which expresses the formal degree of δ_q explicitly as a rational function with rational coefficients on the maximal domain $U_\delta \subset \mathcal{Q}$ to which δ_q extends as a continuous family of irreducible discrete series characters (U_δ is the interior of an explicitly known convex polyhedral cone). At present we do not know how to compute the rational numbers d_δ for each continuous family so our solution is incomplete at this point.

Let us compare our results with the existing literature. An important special case arises when the parameter function q is constant on S, which happens for example when the root system R_0 is irreducible and simply laced. In this case all irreducible representations of $\mathcal{H}(\mathcal{R},q)$ (not only the discrete series) have been classified by

Kazhdan and Lusztig [KL]. This classification is essentially independent of $q \in \mathbb{C}^\times$, except for a few "bad" roots of unity. This work of Kazhdan and Lusztig is of course much more than just a classification of irreducible characters, it actually gives a geometric construction of standard modules of the Hecke algebra for which one can deduce detailed information on the internal structure in geometric terms (e.g. Green functions). The Kazhdan-Lusztig parametrization also yields the classification of the tempered and the discrete series characters.

More recently Lusztig [L3] has given a classification of the irreducibles of the "geometric" graded affine Hecke algebras (with certain unequal parameters) which arise from a cuspidal local system on a unipotent orbit of a Levi subgroup of a given almost simple simply connected complex group ${}^L G$. In [L2] it is shown that such graded affine Hecke algebras can be seen as completions of "geometric" affine Hecke algebras (with certain unequal parameters) formally associated to the above geometric data. On the other hand, let k be a p-adic field and let G be the group of k-rational points of a split adjoint simple group $\mathbf{G}$ over k such that ${}^L G$ is the connected component of the Langlands dual group of $\mathbf{G}$. In [L2] the explicit list of "level 0 arithmetic" affine Hecke algebras is given, i.e. affine Hecke algebras arising as the Hecke algebra of a type (in the sense of [BK]) for a G-inertial equivalence class of a level 0 supercuspidal pair (L, σ) (also see [Mo1], [Mo2]). Remarkably, a case-by-case analysis in [L2] shows that the geometric affine Hecke algebras associated with ${}^L G$ precisely match the level 0 arithmetic affine Hecke algebras arising from G.

The geometric data that Lusztig uses in [L3] to classify the irreducibles of the geometric graded affine Hecke algebras are rather complicated, and the geometry depends on the ratio of the parameters. Our present direct approach, based on deformations in the harmonic analysis of "arithmetic" affine Hecke algebras, gives different and in some sense complementary information (e.g. formal degrees). We refer to [Blo] for examples of affine Hecke algebras arising as Hecke algebras of more general types. We refer to [Lu4] for results and conjectures on the theory of Kazhdan-Lusztig bases of abstract Hecke algebras with unequal parameters.

The techniques in this paper do not give an explicit construction of the discrete series representations. In this direction it is interesting to mention Syu Kato's geometric construction [Kat2] of algebraic families of representations of $\mathcal{H}(C_n^{(1)}, q)$, for *generic* complex parameters q. One would like to understand how Kato's geometric model relates to our continuous families of discrete series representations, which are constructed by analytic methods.

2. Preliminaries and notations

2.1. **Affine Hecke algebras.**

2.1.1. *Root data and affine Weyl groups.* Suppose we are given lattices X, Y in perfect duality $\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{Z}$, and finite subsets $R_0 \subset X$ and $R_0^\vee \subset Y$ with a given a bijection $\vee : R_0 \to R_0^\vee$. Define endomorphisms $r_{\alpha^\vee} : X \to X$ by $r_{\alpha^\vee}(x) = x - x(\alpha^\vee)\alpha$ and $r_\alpha : Y \to Y$ by $r_\alpha(y) = y - \alpha(y)\alpha^\vee$. Then $(R_0, X, R_0^\vee, Y)$ is called a root datum if

(1) for all $\alpha \in R_0$ we have $\alpha(\alpha^\vee) = 2$.
(2) for all $\alpha \in R_0$ we have $r_{\alpha^\vee}(R_0) \subset R_0$ and $r_\alpha(R_0^\vee) \subset R_0^\vee$.

As is well known, it follows that R_0 is a root system in the vector space spanned by the elements of R_0. A *based root datum* $\mathcal{R} = (X, R_0, Y, R_0^\vee, F_0)$ consists of a root datum with a basis $F_0 \subset R_0$ of simple roots.

The (extended) affine Weyl group of $\mathcal{R}$ is the group $W = W_0 \ltimes X$ (where $W_0 = W(R_0)$ is the Weyl group of R_0); it naturally acts on X. We identify $Y \times \mathbb{Z}$ with the set of affine linear, $\mathbb{Z}$-valued functions on X (in this context we usually denote an affine root $a = (\alpha^\vee, n)$ additively as $a = \alpha^\vee + n$). Then the affine Weyl group W acts linearly on the set $Y \times \mathbb{Z}$ via the action $wf(x) := f(w^{-1}x)$. The affine root system R associated to $\mathcal{R}$ is the W-invariant set $R := R_0^\vee \times \mathbb{Z} \subset Y \times \mathbb{Z}$. The basis F_0 of simple roots induces a decomposition $R = R_+ \cup R_-$ with $R_+ := R_{0,+}^\vee \times \{0\} \cup R_0 \times \mathbb{N}$ and $R_- = -R_+$. It is easy to see that R_+ has a bases of affine roots F consisting of the set $F_0^\vee \times \{0\}$ supplemented by the set of affine roots of the form $a = (\alpha^\vee, 1)$ where $\alpha^\vee \in R_0^\vee$ runs over the set of minimal coroots. The set F is called the set of affine simple roots. Every W-orbit $Wa \subset R$ with $a \in R$ meets the set F of affine simple roots. We denote by $\tilde{F}$ the set of intersections of the W-orbits in R with F.

To an affine roots $a = (\alpha^\vee, n)$ we associate an affine reflection $r_a : X \to X$ by $r_a(x) = x - a(x)\alpha$. We have $r_a \in W$ and $wr_aw^{-1} = r_{wa}$. Hence the subgroup $W^a \subset W$ generated by the affine reflections r_a with $a \in R$ is normal. The normal subgroup W^a has a Coxeter presentation (W^a, S) with respect to the set of Coxeter generators $S = \{r_a \mid a \in F\}$. We call S the set of affine simple reflections. We call two elements $s, t \in S$ equivalent if they are conjugate to each other inside W. We put $\tilde{S}$ for the set of equivalence classes in S. The set $\tilde{S}$ is in natural bijection with the set $\tilde{F}$.

We define a length function $l : W \to \mathbb{Z}_+$ by $l(w) := |w^{-1}(R_-) \cap R_+|$. The set $\Omega := \{w \in W \mid l(w) = 0\}$ is a subgroup of W. Since W^a acts simply transitively on the set of positive systems of affine roots it is clear that $W = W^a \rtimes \Omega$. Notice that if we put $X^+ = \{x \in X \mid x(\alpha^\vee) \geq 0 \ \forall \alpha \in F_0\}$ and $X^- = -X^+$ then the sublattice $Z = X^+ \cap X^- \subset X$ is the center of W. It is clear that Z acts trivially on R and in particular, we have $Z \subset \Omega$. We have $\Omega \approx W/W^a \approx X/Q(R_0)$ where $Q(R_0)$ denotes the root lattice of the root system R_0. It follows easily that Ω/Z is finite. We call $\mathcal{R}$ semisimple if $Z = 0$. By the above $\mathcal{R}$ is semisimple iff Ω is finite.

2.1.2. *The generic affine Hecke algebra and its specializations.* We introduce invertible, commuting indeterminates $v([s])$ where $[s] \in \tilde{S}$. Let $\Lambda = \mathbb{C}[v([s])^{\pm 1} : [s] \in \tilde{S}]$. If $s \in S$ then we define $v(s) := v([s])$. The following definition is in fact a theorem (this result goes back to Tits):

Definition 2.1. *There exists a unique associative, unital Λ-algebra $\mathcal{H}_\Lambda(\mathcal{R})$ which has a Λ-basis $\{N_w\}_{w \in W}$ parametrized by $w \in W$, satisfying the relations*

(1) $N_w N_{w'} = N_{ww'}$ *for all* $w, w' \in W$ *such that* $l(ww') = l(w) + l(w')$.
(2) $(N_s - v(s))(N_s + v(s)^{-1}) = 0$ *for all* $s \in S$.

The algebra $\mathcal{H}_\Lambda(\mathcal{R})$ is called the generic affine Hecke algebra with root datum $\mathcal{R}$.

We put $\mathcal{Q}_c = \mathcal{Q}(\mathcal{R})_c$ for the complex torus of homomorphisms $\Lambda \to \mathbb{C}$. We equip the torus $\mathcal{Q}$ with the analytic topology. Given a homomorphism $q \in \mathcal{Q}_c$ we define a specialization[2] $\mathcal{H}(\mathcal{R}, q)$ of the generic algebra as follows (with $\mathbb{C}_q$ the Λ-module

[2]This is not compatible with the conventions in [O1], [O2], [O3], [OS]! The parameter $q \in \mathcal{Q}$ in the present paper would be called $q^{1/2}$ in these earlier papers.

defined by q):

$$\mathcal{H}(\mathcal{R},q) := \mathcal{H}_\Lambda(\mathcal{R}) \otimes_\Lambda \mathbb{C}_q \tag{5}$$

Observe that the automorphism $\phi_s : \Lambda \to \Lambda$ defined by $\phi_s(v(t)) = v(t)$ if $t \not\sim_W s$ and $\phi_s(v(s)) = -v(s)$ extends to an automorphism of $\mathcal{H}_\Lambda$ by putting $\phi_s(N_t) = N_t$ if $t \not\sim_W s$ and $\phi_s(N_s) = -N_s$. Similarly we have automorphims $\psi_s : \mathcal{H}_\Lambda \to \mathcal{H}_\Lambda$ given by $\psi_s(v(s)) = v(s)^{-1}$, $\psi_s(v(t)) = v(t)$ if $t \not\sim_W s$, $\psi_s(N_s) = -N_s$ and $\psi_s(N_t) = N_t$ if $t \not\sim_W s$. These automorphisms mutually commute and are involutive. Observe that $\phi_s\psi_s$ respects the distinguished basis N_w of $\mathcal{H}_\Lambda$, and the automorphisms ϕ_s and ψ_s individually respect the distinguished basis up to signs.

We write $\mathcal{Q}$ for the set of positive points of $\mathcal{Q}_c$, i.e. points $q \in \mathcal{Q}_r$ such that $q(v(s)) > 0$ for all $s \in S$. Then $\mathcal{Q} \subset \mathcal{Q}_c$ is a real vector group.

There are alternative ways to specify points of $\mathcal{Q}$ which play a role in the spectral theory of affine Hecke algebras (in particular in relation to the Macdonald c-function [Ma1]). In order to explain this we introduce the possibly nonreduced root system $R_{nr} \subset X$ associated to $\mathcal{R}$ as follows:

$$R_{nr} = R_0 \cup \{2\alpha \mid \alpha^\vee \in 2Y \cap R_0^\vee\} \tag{6}$$

We define $R_1 = \{\alpha \in R_{nr} \mid 2\alpha \notin R_{nr}\}$. Then $R_1 \subset X$ is also a reduced root system, and $W_0 = W(R_0) = W(R_1)$.

We define various functions with values in Λ. First we define a W-invariant function $R \ni a \to v_a \in \Lambda$ by requiring that

$$v_{a+1} = v(s_a) \tag{7}$$

for all simple affine roots $a \in F$. Notice that all generators $v(s)$ of Λ are in the image of this function. Next we define a W_0-invariant function $R_{nr}^\vee \ni \alpha^\vee \to v_{\alpha^\vee} \in \Lambda$ as follows. If $\alpha \in R_0$ we view $\alpha^\vee$ as an element of R, so that $v_{\alpha^\vee}$ has already been defined. If $\alpha = 2\beta$ with $\beta \in R_0$ then we define:

$$v_{\alpha^\vee} = v_{\beta^\vee/2} := v_{\beta^\vee+1}/v_{\beta^\vee} \tag{8}$$

Finally there exists a unique length-multiplicative function $W \ni w \to v(w) \in \Lambda$ such that its restriction to S yields the original assignment $S \ni s \to v(s) \in \Lambda$ of generators of Λ to the W-orbits of simple reflections of W, and $v(\omega) = 1$ for all $\omega \in \Omega$. Here the notion length-multiplicative refers to the property $v(w_1w_2) = v(w_1)v(w_2)$ if $l(w_1w_2) = l(w_1) + l(w_2)$. We remark that with these notations we have

$$v(w) = \prod_{\alpha \in R_{nr,+} \cap w^{-1}R_{nr,-}} v_{\alpha^\vee} \tag{9}$$

for all $w \in W_0$.

A point $q \in \mathcal{Q}$ determines a unique W-invariant function on R with values in $\mathbb{R}_+$ by defining $q_a := q(v_a)$. Conversely such a positive W-invariant function on R determines a point $q \in \mathcal{Q}$. Likewise we define positive real numbers

$$q_{\alpha^\vee} := q(v_{\alpha^\vee}) \tag{10}$$

for $\alpha \in R_{nr}$ and

$$q(w) := q(v(w)) \tag{11}$$

for $w \in W$. In this way the points $q \in \mathcal{Q}$ are in natural bijection with the set of W_0-invariant positive functions on $R_{nr}^\vee$ and also with the set of positive length-multiplicative functions on W which restrict to 1 on Ω.

Definition 2.2. *If $\mathcal{R}$ is simple and $X = P(R_1)$ (the weight lattice of R_1) we call $\mathcal{H}(\mathcal{R}, q)$ of type $R_1^{(1)}$. This includes the simple 3-parameter case $C_n^{(1)}$ with $R_0 = B_n$ and $X = Q(R_0)$.*

2.1.3. *The Bernstein presentation and the center.* The length function $l : W \to \mathbb{Z}_{\geq 0}$ restricts to a homomorphism of monoids on X^+. Hence the map $X^+ \to \mathcal{H}_\Lambda^\times$ defined by $x \to N_x$ is an homomorphism of monoids too. It has a unique extension to a group homomorphism $\theta : X \to \mathcal{H}_\Lambda^\times$ which we denote by $x \to \theta_x$. We denote by $\mathcal{A}_\Lambda \subset \mathcal{H}_\Lambda$ the commutative subalgebra of $\mathcal{H}_\Lambda$ generated by the elements θ_x with $x \in X$. Let $\mathcal{H}_{\Lambda,0} = \mathcal{H}_\Lambda(W_0, S_0)$ be the Hecke subalgebra (of finite rank over the algebra Λ) corresponding to the Coxeter group (W_0, S_0). We have the following important result due to Bernstein-Zelevinski (unpublished) and Lusztig ([L1]):

Theorem 2.3. *The multiplication map defines an isomorphism of $\mathcal{A}_\Lambda - \mathcal{H}_{\Lambda,0}$-modules $\mathcal{A}_\Lambda \otimes \mathcal{H}_{\Lambda,0} \to \mathcal{H}_\Lambda$ and an isomorphism of $\mathcal{H}_{\Lambda,0} - \mathcal{A}_\Lambda$-modules $\mathcal{H}_{\Lambda,0} \otimes \mathcal{A}_\Lambda \to \mathcal{H}_\Lambda$. The algebra structure on $\mathcal{H}_\Lambda$ is determined by the cross relation (with $x \in X$, $\alpha \in F_0$, $s = r_{\alpha^\vee}$, and $s' \in S$ is a simple reflection such that $s' \sim_W r_{\alpha^\vee+1}$):*

$$\theta_x N_s - N_s \theta_{s(x)} = \left((v(s) - v(s)^{-1}) + (v(s') - v(s')^{-1})\theta_{-\alpha}\right) \frac{\theta_x - \theta_{s(x)}}{1 - \theta_{-2\alpha}} \tag{12}$$

(Note that if $s' \not\sim_W s$ then $\alpha^\vee \in 2R_0^\vee$, which implies $x - s(x) \in 2\mathbb{Z}\alpha$ for all $x \in X$. This guarantees that the right hand side of (12) is always an element of $\mathcal{A}_\Lambda$).

Corollary 2.4. *The center $\mathcal{Z}_\Lambda$ of $\mathcal{H}_\Lambda$ is the algebra $\mathcal{Z}_\Lambda = \mathcal{A}_\Lambda^{W_0}$. For any $q \in \mathcal{Q}_c$ the center of $\mathcal{H}(\mathcal{R}, q)$ is equal to the subalgebra $\mathcal{Z} = \mathcal{A}^{W_0} \subset \mathcal{H}(\mathcal{R}, q)$.*

In particular $\mathcal{H}_\Lambda$ is a finite type algebra over its center $\mathcal{Z}_\Lambda$, and similarly $\mathcal{H}(\mathcal{R}, q)$ is a finite type algebra over its center $\mathcal{Z}$. The simple modules over these algebras are finite dimensional complex vector spaces. The primitive spectrum $\widehat{\mathcal{H}}_\Lambda$ is a topological space which comes equipped with a finite continuous and closed map

$$cc_\Lambda : \widehat{\mathcal{H}}_\Lambda \to \widehat{\mathcal{Z}}_\Lambda = W_0 \backslash T \times \mathcal{Q}_c \tag{13}$$

to the complex affine variety associated with the unital complex commutative algebra $\mathcal{Z}_\Lambda$. The map cc_Λ is called the central character map. Similarly, we have central character maps

$$cc_q : \widehat{\mathcal{H}(\mathcal{R}, q)} \to \widehat{\mathcal{Z}} \tag{14}$$

for all $q \in \mathcal{Q}_c$.

We put $T^{alg} = \mathrm{Hom}(X, \mathbb{C}^\times)$, the complex torus of characters of the lattice X equipped with the Zariski topology. This torus has a natural W_0-action. We have $\widehat{\mathcal{Z}} = W_0 \backslash T^{alg}$ (the categorical quotient).

2.1.4. *Two reduction theorems.* The study of the simple modules over $\mathcal{H}(\mathcal{R}, q)$ is simplified by two reduction theorems which are much in the spirit of Lusztig's reduction theorems in [L1]. The first of these theorems reduces to the case of simple modules whose central character is a W_0-orbit of characters of X which are positive on the sublattice of X spanned by R_1 (see the explanation below). The second theorem reduces the study of simple modules of $\mathcal{H}(\mathcal{R}, q)$ with a positive central character in the above sense to the study of simple modules of an associated degenerate affine Hecke algebra with real central character. These results will be useful for our study of the discrete series characters.

First of all a word about terminology. The complex torus T has a polar decomposition $T = T_v T_u$ with $T_v = \mathrm{Hom}(X, \mathbb{R}_{>0})$ and $T_u = \mathrm{Hom}(X, S^1)$. The polar decomposition is the exponentiated form of the decomposition of the tangent space $V = \mathrm{Hom}(X, \mathbb{C})$ of T at $t = e$ as a direct sum $V = V_r \oplus iV_r$ of real subspaces where $V_r = \mathrm{Hom}(X, \mathbb{R})$. The vector group T_v is called the group of positive characters and the compact torus T_u is called the group of unitary characters. This polar decomposition is compatible with the action of W_0 on T. We call the W_0-orbits of points in T_v "positive" and the W_0-orbits of points in T_u "unitary". In this sense can we speak of the subcategory of finite dimensional $\mathcal{H}(\mathcal{R}, q)$-modules with positive central character[3] which is a subcategory that plays an important special role.

Definition 2.5. *Let $\mathcal{R}$ be a root datum and let $q \in \mathcal{Q} = \mathcal{Q}(\mathcal{R})$. For $s \in T_u$ we define $R_{s,0} = \{\alpha \in R_0 \mid r_\alpha(s) = s\}$. Let $R_{s,1} \subset R_1$ be the set of inmultiplicable roots corresponding to $R_{s,0}$. One checks that*

$$R_{s,1} = \{\beta \in R_1 \mid \beta(s) = 1\} \tag{15}$$

Let $R_{s,1,+} \subset R_{s,1}$ be the unique system of positive roots such that $R_{s,1,+} \subset R_{1,+}$, and let $F_{s,1}$ be the corresponding basis of simple roots of $R_{s,1}$. Then the isotropy group $W_s \subset W_0$ of s is of the form

$$W_s = W(R_{s,1}) \rtimes \Gamma_s \tag{16}$$

where $\Gamma_s = \{w \in W_s \mid w(R_{s,1,+}) = R_{s,1,+}\}$ is a group acting through diagram automorphisms on the based root system $(R_{s,1}, F_{s,1})$.

We form a new root datum $\mathcal{R}_s = (X, R_{s,0}, Y, R_{s,0}^\vee, F_{s,0})$ and observe that $R_{nr,s} \subset R_{nr}$. Hence we can define a surjective map $\mathcal{Q}(\mathcal{R}) \to \mathcal{Q}(\mathcal{R}_s)$ (denoted by $q \to q_s$) by restriction of the corresponding parameter function on $R_{nr}^\vee$ to $R_{nr,s}^\vee$.

Let $t = cs \in T_v T_u$ be the polar decomposition of an element $t \in T$. We define $W_0(t) \subset W_s$ for the subgroup defined by

$$W_0(t) := \{w \in W_s \mid wt \in W(R_{s,1})t\} \tag{17}$$

Observe that $W_0(t)$ is the semidirect product $W_0(t) = W(R_{s,1}) \rtimes \Gamma(t)$ where

$$\Gamma(t) = \Gamma_s \cap W_0(t) \tag{18}$$

Let $M_{W_0 t} \subset \mathcal{Z}$ denote maximal ideal of $\mathcal{A}$ of elements vanishing at $W_0 t \subset T$, and let $\overline{\mathcal{Z}}$ be the $M_{W_0 t}$-adic completion of $\mathcal{Z}$. We define

$$\overline{\mathcal{A}} = \mathcal{A} \otimes_{\mathcal{Z}} \overline{\mathcal{Z}} \tag{19}$$

By the Chinese remainder theorem we have

$$\overline{\mathcal{A}} = \bigoplus_{t' \in W_0 t} \overline{\mathcal{A}}_{t'} \tag{20}$$

where $\overline{\mathcal{A}}_{t'}$ denotes the formal completion of $\mathcal{A}$ at $t' \in T$. Let $1_{t'}$ denote the unit of the summand $\overline{\mathcal{A}}_{t'}$ in this direct sum decomposition. We consider the formal completion

$$\overline{\mathcal{H}}(\mathcal{R}, q) = \mathcal{H}(\mathcal{R}, q) \otimes_{\mathcal{Z}} \overline{\mathcal{Z}} \tag{21}$$

[3] In several prior publications [HO1], [HO2], [O1], [O2], [O3] the central characters in $W_0 \backslash T_v$ were referred to as "real central characters", where "real" should be understood as "infinitesimally real". In the present paper however we change the terminology and speak of "positive central characters" instead.

On the other hand, we consider the affine Hecke algebra $\mathcal{H}(\mathcal{R}_s, q_s)$ and its commutative subalgebra $\mathcal{A}_s$ (as defined before when discussing the Bernstein basis) and center $\mathcal{Z}_s = \mathcal{A}_s^{W(R_{s,1})}$. Let $m_{W(R_{s,1})t}$ be the maximal ideal in $\mathcal{Z}_s$ of elements vanishing at the orbit $W(R_{s,1})t = sW(R_{s,1})c$; let $\overline{\mathcal{Z}_s}$ and $\overline{\mathcal{H}}(\mathcal{R}_s, q_s)$ be the corresponding formal completions as before.

The group $\Gamma(t)$ acts on $\overline{\mathcal{H}}(\mathcal{R}_s, q_s)$ and on its center $\overline{\mathcal{Z}_s}$. We note that there exists a canonical isomorphism

$$\overline{\mathcal{Z}} \to \overline{\mathcal{Z}_s}^{\Gamma(t)} \tag{22}$$

As before we define a localization

$$\overline{\mathcal{H}}(\mathcal{R}_s, q_s) = \mathcal{H}(\mathcal{R}_s, q_s) \otimes_{\mathcal{Z}_s} \overline{\mathcal{Z}_s} \tag{23}$$

Let $e_t \in \overline{\mathcal{A}} \subset \overline{\mathcal{H}}(\mathcal{R}, q)$ be the idempotent defined by

$$e_t = \sum_{t' \in W(R_{s,1})t} 1_{t'} \tag{24}$$

Theorem 2.6. *("First reduction Theorem" (see* [L1, Theorem 8.6]*)) Let $q \in \mathcal{Q}$ and let $t = cs$ be the polar decomposition of an element $t \in T$. Let n be the cardinality of the orbit W_0t divided by the cardinality of the orbit $W(R_{s,1})t$. Using the notations introduced above, there exists an isomorphism of $\overline{\mathcal{Z}}$-algebras*

$$(\overline{\mathcal{H}}(\mathcal{R}_s, q_s) \rtimes \Gamma(t))_{n\times n} \to \overline{\mathcal{H}}(\mathcal{R}, q) \tag{25}$$

Via this isomorphism the idempotent $e_t \in \overline{\mathcal{H}}(\mathcal{R}, q)$ corresponds to the $n \times n$-matrix with 1 in the upper left corner and 0's elsewhere. Hence the $\overline{\mathcal{Z}}$-algebras $\overline{\mathcal{H}}(\mathcal{R}, q)$ and $\overline{\mathcal{H}}(\mathcal{R}_s, q_s) \rtimes \Gamma(t)$ are Morita equivalent. In particular the set of simple modules U of $\mathcal{H}(\mathcal{R}, q)$ with central character W_0t corresponds bijectively to the set of simple modules V of $\mathcal{H}(\mathcal{R}_s, q_s) \rtimes \Gamma(t)$ with central character $W_0(s)t = W(R_{s,1})t$, where the bijection is given by $U \to e_tU$.

Proof. The proof is a straightforward translation of Lusztig's proof of [L1, Theorem 8.6]. We replace the equivalence relation that Lusztig defines on the orbit W_0t by the equivalence relation induced by the action of $W(R_{s,1})$ (i.e. the equivalence classes are the orbits of $W(R_{s,1})$ in W_0t; in other words, the role of the subgroup $\mathcal{J}\langle v_0\rangle \subset T$ in Lusztig's setup is now played by the vector subgroup T_v). After this change the rest of the proof is identical to Lusztig's proof. □

The second reduction theorem gives a bijection between simple modules of affine Hecke algebra's with central character W_0t satisfying $\alpha(t) > 0$ for all $\alpha \in R_1$ and simple modules of an associated degenerate affine Hecke algebra with a real central character. We first need to define the appropriate notion of the associate degenerate affine Hecke algebra.

Let $\mathcal{R} = (X, R_0, Y, R_0^\vee, F_0)$ be a root datum, let $q \in \mathcal{Q}$, and let $W_0t \in W_0\backslash T$ be a central character such that for all $\alpha \in R_1$ we have $\alpha(t) \in \mathbb{R}_{>0}$. Then the polar decomposition of t has the form $t = uc$ with $u \in T_u$ a W_0-invariant character of X and with $c \in T_v$ a positive character of X. Observe that $\beta(u) = 1$ if $\beta \in R_0 \cap R_1$ and $\beta(u) = \pm 1$ if $\beta \in R_0 \cap \frac{1}{2}R_1$. We define a W_0-invariant real parameter function

$k_u : R_1 \to \mathbb{R}$ by the following prescription. If $\alpha \in R_1$ we put:

$$k_{u,\alpha} = \begin{cases} \log(q_{\alpha^\vee}^2) & \text{if } \alpha \in R_0 \cap R_1 \\ \log(q_{\alpha^\vee}^2 q_{2\alpha^\vee}^4) & \text{if } \alpha = 2\beta \text{ with } \beta \in R_0 \text{ and } \beta(u) = 1 \\ \log(q_{\alpha^\vee}^2) & \text{if } \alpha = 2\beta \text{ with } \beta \in R_0 \text{ and } \beta(u) = -1 \end{cases} \tag{26}$$

Definition 2.7. *We define the degenerate affine Hecke algebra* $\mathbf{H}(R_1, V, F_1, k)$ *associated with the root system* $R_1 \subset V^*$ *where* $V = \mathbb{R} \otimes Y$ *and the parameter function* k *as follows. We put* $P(V)$ *for the polynomial algebra on the vector space* V*. The Weyl group* W_0 *acts on* $P(V)$ *and we denote the action by* $w \cdot f = f^w$*. Then* $\mathbf{H}(R_1, V, F_1, k)$ *is simultaneously a left* $P(V)$*-module and a right* $\mathbb{C}[W_0]$*-module, and as such it has the structure* $\mathbf{H}(R_1, V, F_1, k) = P(V) \otimes \mathbb{C}[W_0]$*. We identify* $P(V) \otimes e \subset \mathbf{H}(R_1, V, F_1, k)$ *with* $P(V)$ *and* $1 \otimes \mathbb{C}[W_0] \subset \mathbf{H}(R_1, V, F_1, k)$ *with* $\mathbb{C}[W_0]$ *so that we may write* fw *instead of* $f \otimes w$ *if* $f \in P(V)$ *and* $w \in W_0$*. The algebra structure structure of* $\mathbf{H}(R_1, V, F_1, k)$ *is uniquely determined by the cross relation (with* $f \in P(V)$*,* $\alpha \in F_1$ *and* $s = s_\alpha \in S_1$*):*

$$fs - sf^s = k_\alpha \frac{f - f^s}{\alpha} \tag{27}$$

It is easy to see that the center of $\mathbf{H}(R_1, V, F_1, k)$ is equal to the algebra $\mathbf{Z} = P(V)^{W_0} \subset \mathbf{H}(R_1, V, F_1, k)$. The vector space $V_c = \mathbb{C} \otimes V$ can be identified with the Lie algebra of the complex torus T. Let $\exp : V_c \to T$ be the corresponding exponential map. It is a W_0-equivariant covering map which restricts to a group isomorphism $V \to T_v$ of the real vector space V to the vector group T_v.

Theorem 2.8. *("Second reduction Theorem" (see* [L1, Theorem 9.3]*)) Let* $\mathcal{R} = (X, R_0, Y, R_0^\vee, F_0)$ *be a root datum with parameter function* $q \in \mathcal{Q} = \mathcal{Q}(\mathcal{R})$*. Let* $V_0 \subset V$ *be the subspace spanned by* $R_0^\vee$*. Given* $t \in T$ *such that* $\alpha(t) > 0$ *for all* $\alpha \in R_1$ *we let* $\xi = \xi_t \in V$ *be the unique element such that* $\alpha(t) = \mathrm{e}^{\alpha(\xi)}$ *for all* $\alpha \in R_1$*. It is easy to see that the map* $t \to \xi = \xi_t$ *is* W_0*-equivariant; in particular the image of* $W_0 t$ *is equal to* $W_0 \xi$*. Let* $t = uc$ *be the polar decomposition of* t*. Then* $u \in T_u$ *is* W_0*-invariant, and we define a* W_0*-invariant parameter function* k_u *on* R_1 *by (26). Let* $\overline{\mathbf{Z}}$ *be the formal completion of the center* $\mathbf{Z}$ *of* $\mathbf{H}(R_1, V, F_1, k_u)$ *at the orbit* $W_0 \xi$*. Let* $\mathbf{P} = P(V)$ *and put* $\overline{\mathbf{P}} = \mathbf{P} \otimes_{\mathbf{Z}} \overline{\mathbf{Z}}$ *and* $\overline{\mathbf{H}}(R_1, V, F_1, k_u) = \mathbf{H}(R_1, V, F_1, k_u) \otimes_{\mathbf{Z}} \overline{\mathbf{Z}}$*. There exist natural compatible isomorphism of algebras* $\overline{Z} \to \overline{\mathbf{Z}}$*,* $\overline{\mathcal{A}} \to \overline{\mathbf{P}}$ *and* $\overline{\mathcal{H}}(\mathcal{R}, q) \to \overline{\mathbf{H}}(R_1, V, F_1, k_u)$*.*

Proof. This is a straightforward translation of the proof of [L1, Theorem 9.3]. □

Corollary 2.9. *The set of simple modules of* $\mathcal{H}(\mathcal{R}, q)$ *with central character* $W_0 t$ *(satisfying the above condition that* $\alpha(t) > 0$ *for all* $\alpha \in R_1$*) and the set of simple modules of* $\mathbf{H}(R_1, V, F_1, k_u)$ *with central character* $W_0 \xi$ *(as described in Theorem 2.8) are in natural bijection.*

Combining the two reduction theorems we finally obtain the following result (see [L1, Section 10]):

Corollary 2.10. *For all* $s \in T_u$ *the center of* $\mathbf{H}(R_{s,1}, V, F_{s,1}, k_s) \rtimes \Gamma(t)$ *is equal to* $\mathbf{Z}^{\Gamma(t)}$*. If* $t \in T$ *is arbitrary with polar decomposition* $t = sc$*, then the set of simple modules of* $\mathcal{H}(\mathcal{R}, q)$ *with central character* $W_0 t$ *is in natural bijection with the set of simple modules of* $\mathbf{H}(R_{s,1}, V, F_{s,1}, k_s) \rtimes \Gamma(t)$ *with the real central character* $W_s \xi$*. Here* $\xi \in V$ *is the unique vector in the real span of* $R_{s,1}^\vee$ *such that* $\alpha(t) = \mathrm{e}^{\alpha(\xi)}$ *for*

all $\alpha \in R_{s,1}$, k_s is the real parameter function on $R_{s,1}$ associated to q_s described by (26), and $\Gamma(t)$ is the group defined by (18).

2.2. **Harmonic analysis for affine Hecke algebras.**

2.2.1. *The Hilbert algebra structure of the Hecke algebra.* Let $\mathcal{R}$ be a based root datum and $q \in \mathcal{Q}$ a *positive* parameter function for $\mathcal{R}$. We turn $\mathcal{H} = \mathcal{H}(\mathcal{R}, q)$ into a $*$-algebra using the conjugate lineair anti-involution $* : \mathcal{H} \to \mathcal{H}$ defined by $N_w^* = N_{w^{-1}}$. We define a trace $\tau : \mathcal{H} \to \mathbb{C}$ by $\tau(N_w) = \delta_{w,e}$. This defines a Hermitian form $(x, y) := \tau(x^* y)$ with respect to which the basis N_w is orthonormal. In particular $(\cdot, \cdot)$ is positive definite. In fact it is easy to show [O1] that this Hermitian inner product defines the structure of a Hilbert algebra on $\mathcal{H}$. Let $L^2(\mathcal{H})$ be the Hilbert space completion of $\mathcal{H}$ and let $\mathfrak{C} := C_r^*(\mathcal{H}) \subset B(L^2(\mathcal{H}))$ be the C^*-algebra completion of the image of $\mathcal{H}$ inside the algebra of bounded linear operators on $L^2(\mathcal{H})$. This C^*-algebra is called the reduced C^*-algebra of $\mathcal{H}$. It is not hard to show that $\mathfrak{C}$ is unital, separable and liminal, which implies that the spectrum $\hat{\mathfrak{C}}$ of $\mathfrak{C}$ is a compact T_1 space with countable base which contains an open dense Hausdorff subset. The trace τ extends to a finite tracial state τ on $\mathfrak{C}$. In this situation (see [O1, Theorem 2.25]) there exists a unique positive Borel measure μ_{Pl} on $\hat{\mathfrak{C}}$ such that for all $h \in \mathcal{H}$:

$$\tau = \int_{\hat{\mathfrak{C}}} \chi_\pi d\mu_{Pl}(\pi) \tag{28}$$

Since τ is faithful it follows that the support of μ_{Pl} is equal to $\hat{\mathfrak{C}}$.

Definition 2.11. *We call the measure μ_{Pl} the Plancherel measure of $\mathcal{H}$.*

Definition 2.12. *An irreducible $*$-representation (V, π) of the involutive algebra $\mathcal{H}$ is called a discrete series representation of $\mathcal{H}$ if (V, π) extends to a representation (also denoted (V, π)) of $\mathfrak{C}$ which is equivalent to a subrepresentation of the left regular representation of $\mathfrak{C}$ on $L^2(\mathcal{H})$. In this case the finite trace χ_π defined by $\chi_\pi(x) = \mathrm{Tr}_V(\pi(x))$ is called an irreducible discrete series character.*

We have seen that an irreducible representation (V, π) of $\mathcal{H}$ is finite dimensional. In particular its character χ_π is a well defined linear functional on $\mathcal{H}$. We call χ_π an irreducible character of $\mathcal{H}$. Clearly the character of a finite dimensional representation of $\mathcal{H}$ only depends on the equivalence class of the underlying representation. The irreducible characters of a set of mutually inequivalent irreducible representations of $\mathcal{H}$ are linearly independent (see [O1, Corollary 2.11]). Hence the equivalence class of a finite dimensional semisimple representation is completely determined by its character.

Definition 2.13. *We denote by $\Delta(\mathcal{R}, q)$ the set of irreducible discrete series characters of $\mathcal{H}(\mathcal{R}, q)$. For each irreducible character $\chi \in \Delta(\mathcal{R}, q)$ we choose and fix an irreducible discrete series representation (V, δ) of $\mathcal{H}$ such that $\chi = \chi_\delta$ (by abuse of language we will often identify the set of irreducible discrete series characters and (the chosen set of representatives of) the set of equivalence classes of irreducible discrete series representations).*

The following criterion for an irreducible representation (V, π) of $\mathcal{H}$ to belong to the discrete series follows from a general result of Dixmier (see [O1]):

Corollary 2.14. *(V, π) is a discrete series representation iff $\mu_{Pl}(\{\pi\}) > 0$.*

Corollary 2.15. *(see* [O1, Proposition 6.10]*) There is a 1-1 correspondence between the set of irreducible discrete series characters χ_δ and the set of primitive central Hermitian idempotents $e_\delta \in \mathfrak{C}$ of finite rank. The correspondence is such that $\tau(e_\delta x) = \mu_{Pl}(\{\delta\})\chi_\delta(x)$ for all $x \in \mathcal{H}$.*

Corollary 2.16. *(see* [O1, Proposition 6.10]*) (V, π) is a discrete series representation iff $\{[\pi]\} \subset \hat{\mathfrak{C}}$ is a component of $\hat{\mathfrak{C}}$. In particular, the number of irreducible discrete series characters is finite.*

2.2.2. *The Schwartz algebra.* We define a nuclear Fréchet algebra $\mathcal{S} = \mathcal{S}(\mathcal{R}, q)$ (the Schwartz algebra) which plays a pivotal role in the spectral theory of the trace τ on $\mathcal{H}$.

Definition 2.17. *We choose once and for all a rational, W_0-invariant inner product $\langle \cdot, \cdot, \rangle$ on the vector space $V^* := \mathbb{Q} \otimes X$.*

Let V_0^* be the rational vector space spanned by R_0. Its orthocomplement is the rational vector space $V_Z^* = \mathbb{Q} \otimes Z$ spanned by the center Z of W. Given $\phi \in V^*$ we decompose $\phi = \phi_0 + \phi_Z$ with respect to the orthogonal decomposition $V^* = V_0 \oplus V_Z$.

Definition 2.18. *We define a norm $\mathcal{N} : W \to \mathbb{R}_+$ on W as follows: if $w \in W$ we put*

$$\mathcal{N}(w) = l(w) + \|w(0)_Z\| \tag{29}$$

Next we define seminorms $p_n : \mathcal{H} \to \mathbb{R}_+$ on $\mathcal{H}$ by

$$p_n(h) := \max_{w \in W}(1 + \mathcal{N}(w))^n |(N_w, h)| \tag{30}$$

Definition 2.19. *The Schwartz algebra $\mathcal{S}$ of $\mathcal{H}$ is the completion of $\mathcal{H}$ with respect to the system of seminorms p_n with $n \in \mathbb{N}$.*

Theorem 2.20. *(*[O1], [Sol]*) The completion $\mathcal{S}$ is a Fréchet algebra which is continuously and densely embedded in $\mathfrak{C}$.*

Remark 2.21. *The Fréchet algebra $\mathcal{S}$ is independent of the choice made in Definition 2.17. $\mathcal{S}$ is also independent of $q \in \mathcal{Q}$ as a Fréchet space.*

Definition 2.22. *A finite dimensional representation of $\mathcal{H}$ is called tempered if it has a continuous extension to $\mathcal{S}$.*

The Fréchet algebra structure of $\mathcal{S}$ depends on $q \in \mathcal{Q}$. The basic theorem 2.20 was first proven in [O1] using some qualitative analysis on the spectrum of $\mathfrak{C}$; the proof in [Sol] is more direct and uses an elementary but nontrivial result due to Lusztig on the multiplication table of $\mathcal{H}$ with respect to the basis N_w. The latter proof also reveals the following important fact with respect to the dependence of $q \in \mathcal{Q}$:

Theorem 2.23. *(see* [Sol, Proposition 5.9, Corollary 5.10]*) The dense subalgebra $\mathcal{S} \subset \mathfrak{C}$ is closed for holomorphic calculus (also see* [DO, Corollary 5.9]*). The holomorphic calculus is continuous on $\mathcal{S} \times \mathcal{Q}$ in the following sense. Let $U \subset \mathbb{C}$ be an open set. The set $V_U \subset \mathcal{S} \times \mathcal{Q}$ defined by $V_U = \{(x, q) \mid \mathrm{Sp}(x, q) \subset U\}$ is open. For any holomorphic function $f : U \to \mathbb{C}$ the map $V_U \ni (x, q) \to f(x, q) \in \mathcal{S}$ is continuous.*

The following result shows the fundamental role of $\mathcal{S}$ for the spectral theory of τ:

Theorem 2.24. *([DO, Corollary 4.4]) The support of μ_{Pl} consists precisely of the set of equivalence classes of irreducible tempered representations of $\mathcal{H}$.*

In particular the discrete series representations are tempered. There are various characterizations of tempered representations and of discrete series representations. Casselman's criterion states that:

Theorem 2.25. *(Casselman's criterion, see [O1, Lemma 2.22]) Let (V,δ) be an irreducible representation of $\mathcal{H}$. The following are equivalent:*

1. *(V,δ) is a discrete series representation of $\mathcal{H}$.*
2. *All matrix coefficients of (V,δ) belong to $L^2(\mathcal{H})$.*
3. *The character χ_δ of (V,δ) belongs to $L^2(\mathcal{H})$.*
4. *All generalized $\mathcal{A}$-weights $t\in T$ in V satisfy: $|x(t)|<1$ for all $x\in X^+\backslash\{0\}$.*
5. *For every matrix coefficient m of δ there exist constants $C,\epsilon>0$ such that $|m(N_w)|<C\mathrm{e}^{-\epsilon\mathcal{N}(w)}$ for all $w\in W$.*
6. *The character χ_δ of (V,δ) belongs to $\mathcal{S}$.*

Corollary 2.26. *An irreducible representation (V,δ) of $\mathcal{H}$ is an irreducible discrete series representation iff (V,δ) is afforded by a central primitive idempotent $e_\delta\in\mathcal{S}$ of $\mathcal{S}$ (see Corollary 2.15).*

Corollary 2.27. *The set $\Delta(\mathcal{R},q)$ is nonempty only if $\mathcal{R}$ is semisimple.*

Casselman's criterion for discrete series in terms of the generalized $\mathcal{A}$-weights can be transposed to define the notion of discrete series modules over a crossed product $\mathbf{H}(R_1,V,F_1,k)\rtimes\Gamma$ of a degenerate affine Hecke algebra $\mathbf{H}(R_1,V,F_1,k)$ with a real parameter function k and a finite group Γ acting by diagram automorphisms of (R_1,F_1) (thus a simple module (U,δ) is a discrete series representation iff the generalized $\mathbf{P}$-weights in U are in the interior of the antidual cone ($\subset V$) of the simplicial cone spanned by F_1). It is clear that this definition is compatible with the bijections afforded by the two reduction theorems (Theorem 2.6 and Theorem 2.8). Hence we obtain from Corollary 2.10:

Corollary 2.28. *Let $t\in T$ with polar decomposition $t=sc$. The set Δ_{W_0t} of equivalence classes of irreducible discrete series representations of $\mathcal{H}(\mathcal{R},q)$ with central character W_0t is in natural bijection with the set of equivalence classes of irreducible discrete series representations of $\mathbf{H}(R_{s,1},V,F_{s,1},k_s)\rtimes\Gamma(t)$ with the real central character $W_s\xi$. Here $\xi\in V$ is the unique vector in the real span of $R_{s,1}^\vee$ such that $\alpha(t)=\mathrm{e}^{\alpha(\xi)}$ for all $\alpha\in R_{s,1}$, k_s is the real parameter function on $R_{s,1}$ described by (26), and $\Gamma(t)$ is the group of diagram automorphisms of $(R_{s,1},F_{s,1})$ of (18).*

Corollary 2.29. *If $\Delta_{W_0t}\neq\emptyset$ then the polar decomposition $t=sc$ of t has the property that $R_{s,1}\subset R_1$ is a root subsystem of maximal rank.*

If $s=u\in T_u$ is W_0-invariant (i.e. if $\alpha(u)=1$ for all $\alpha\in R_1$) then we obtain from Corollary 2.28:

Corollary 2.30. *Let $u\in T_u$ be W_0-invariant, and let $c\in T_v$. There is a natural bijection between the set $\Delta(\mathcal{R},q)_{uW_0c}$ of irreducible discrete series characters of $\mathcal{H}(\mathcal{R},q)$ with central character of the form $uW_0c\subset W_0\backslash T$ and the set of irreducible discrete series characters of $\mathbf{H}(R_1,V,F_1,k_u)$ with the real infinitesimal central character $W_0\log(c)$.*

It is not hard to show that the central character of an irreducible discrete series character of $\mathbf{H}(R_1, V, F_1, k_u)$ is real (see [Slo1, Lemma 1.3.4]). Hence the previous corollary in particular says that:

Corollary 2.31. *Let $u \in T_u$ be W_0-invariant. There is a natural bijection between the set $\Delta^u(\mathcal{R}, q)$ of irreducible discrete series characters of $\mathcal{H}(\mathcal{R}, q)$ with a central character of the form uW_0c with $c \in T_v$ on the one hand, and the set $\Delta^{\mathbf{H}}(R_1, V, F_1, k)$ of irreducible discrete series characters of $\mathbf{H}(R_1, V, F_1, k_u)$ on the other hand. In this bijection the correspondence of the central characters is as described above.*

We can use Corollary 2.28 to reduce the general classification problem of the irreducible discrete series characters of $\mathcal{H}(\mathcal{R}, q)$ for any semisimple root datum $\mathcal{R}$ to the case of discrete series characters of a degenerate affine Hecke algebra as well, but we have to pay the price of having to deal with crossed products by certain groups of diagram automorphisms. In order to deal with the crossed products one has to resort to Clifford theory (cf. [RR]).

Corollary 2.26 gives us yet another characterization of the irreducible discrete series representations:

Theorem 2.32. *Let (V, δ) be a simple module over $\mathcal{H}$. Equivalent are:*

(1) *(V, δ) is a discrete series representation of $\mathcal{H}$.*
(2) *(V, δ) extends to a projective $\mathcal{S}$-module.*

2.2.3. *The Euler-Poincaré pairing and elliptic characters.* We recall the main result of [OS]:

Theorem 2.33. *The affine Hecke algebra $\mathcal{H} = \mathcal{H}(\mathcal{R}, q)$ has global homological dimension equal to the rank of X. If U, V are finite dimensional tempered $\mathcal{H}$-modules then for all i we have $\mathrm{Ext}^i_{\mathcal{H}}(U, V) \cong \mathrm{Ext}^i_{\mathcal{S}}(U, V)$.*

Define the Euler-Poincaré pairing on the (complexified) Grothendieck group $G(\mathcal{H})$ of finite dimensional virtual characters by sesquilinear extension from the formula

$$\mathrm{EP}(U, V) := \sum_{i=0}^{\infty} (-1)^i \dim(\mathrm{Ext}^i_{\mathcal{H}}(U, V)) \tag{31}$$

It can be seen that this defines a Hermitian positive semidefinite pairing on $G(\mathcal{H})$ ([OS, Theorem 3.5]). The above result combined with Theorem 2.32 implies that:

Corollary 2.34. *The irreducible discrete series characters of $\mathcal{H}$ form an orthonormal set with respect to* EP *and are orthogonal to all irreducible tempered characters that are not in the discrete series.*

Another crucial result of [OS] says that EP factors through the quotient $\mathrm{Ell}(\mathcal{H})$ of $G(\mathcal{H})$ by the subspace spanned by all the properly induced finite dimensional tempered characters. Then $\mathrm{Ell}(\mathcal{H})$ is a finite dimensional $\mathcal{Z}$-module, equipped with a positive semidefinite Hermitian pairing EP with respect to which elements with a disjoint support on $W_0 \backslash T$ are orthogonal.

There exists a scaling map $\sigma_0 : G(\mathcal{H}) \to G(W)$ (see [OS, Theorem 1.7]) which descends to a map $\tilde{\sigma}_0 : \mathrm{Ell}(\mathcal{H}) \to \mathrm{Ell}(W)$. The finite dimensional $\mathcal{Z}$-module $\mathrm{Ell}(W)$ can be described completely explicitly in terms of the elliptic characters of the isotropy groups W_t (with $t \in T$) for the action of W_0 on T. The pairing EP on $\mathrm{Ell}(W)$ can be described in these terms as well, and it turns out that EP is positive definite on

$\mathrm{Ell}(W)$ (for all these results, consult [OS, Chapter 3]). It turns out that $\mathrm{Ell}(W)$ is nonzero only if $\mathcal{R}$ is semisimple, and that the support of $\mathrm{Ell}(W)$ as a $\mathcal{Z}$-module is contained in the the set of orbits $W_0 s$ such that $R_{s,1} \subset R_1$ is of maximal rank. From [OS] we have:

Theorem 2.35. (1) *The map* $\tilde{\sigma}_0 : \mathrm{Ell}(\mathcal{H}) \to \mathrm{Ell}(W)$ *is isometric with respect to* EP.

(2) *For all* $t \in T$ *we have* $\tilde{\sigma}_0(\mathrm{Ell}_{W_0 t}(\mathcal{H})) \subset \mathrm{Ell}_{W_0 s}(W)$*, where* $t = sc$ *with* $s \in T_u$ *and* $c \in T_v$ *is the polar decomposition of* t.

Combined with Corollary 2.34 we obtain the following upper bounds for the number of discrete series characters.

Corollary 2.36. *If* $s \in T_u$ *then* W_s *denoted the isotropy group of* s *in* W_0*. We call* $w \in W_s$ *elliptic if* s *is an isolated fixed point of* w*. Let* $\mathrm{ell}(W_s)$ *be the number of conjugacy classes of* W_s *consisting of elliptic elements of* W_s*. For* $s \in T_u$ *we denote by* $\Delta^s(\mathcal{R}, q) \subset \Delta(\mathcal{R}, q)$ *the subset consisting of the irreducible discrete series characters of* $\mathcal{H}(\mathcal{R}, q)$ *whose central characters are* W_0*-orbits which are contained in the set* $W_0 s T_v$*. Then* $|\Delta^s(\mathcal{R}, q)| \leq \mathrm{ell}(W_s)$*.*

2.3. **The central support of tempered characters.** In this section deformations in the parameters q of the Hecke algebra play a fundamental role. Let us fix some notations and basic structures. Recall that we attach to a based root datum $\mathcal{R} = (X, R_0, Y, R_0^\vee, F_0)$ in a canonical way a parameter space $\mathcal{Q} = \mathcal{Q}(\mathcal{R})$. This parameter space is itself a vector group, defined as the space of length multiplicative functions $q : W \to \mathbb{R}_+$ with the additional requirement that $q|_\Omega = 1$.

The following proposition is useful in order to reduce statements about residual points to the case of simple root data.

Proposition 2.37. *Let* $\mathcal{R} = (X, R_0, Y, R_0^\vee, F_0)$ *be a semisimple based root datum.*

(i) *Let* $R_0 = R_0^{(1)} \times \cdots \times R_0^{(m)}$ *be the decomposition of* R_0 *in irreducible components. We denote by* $X^{(i)}$ *be the projection of the lattice* X *onto* $\mathbb{R}R_0^{(i)}$*, and we define* $\mathcal{R}^{(i)} = (X^{(i)}, R_0^{(i)}, Y^{(i)}, (R_0^{(i)})^\vee, F_0^{(i)})$ *and* $\mathcal{R}' = \mathcal{R}^{(1)} \times \cdots \times \mathcal{R}^{(m)}$*. Then the natural inclusion* $X \hookrightarrow X'$ *defines an isogeny* $\psi : \mathcal{R} \to \mathcal{R}'$ *and if* $\mathcal{Q}^{(i)}$ *denotes be the parameter space of the root datum* $\mathcal{R}^{(i)}$ *then* ψ *yields a natural identification* $\mathcal{Q}(\mathcal{R}) = \mathcal{Q}(\mathcal{R}') = \mathcal{Q}^{(1)} \times \cdots \times \mathcal{Q}^{(m)}$*.*

(ii) *We replace* X *by the lattice* $X^{max} = P(R_1)$*, the weight lattice of* R_1 *and denote the resulting root datum by* $\mathcal{R}^{max}$*. Then* $\mathcal{R}^{max}$ *is a direct product of irreducible root data and there exists an isogeny* $\psi : \mathcal{R} \to \mathcal{R}^{max}$ *which yields a natural identification* $\mathcal{Q}(\mathcal{R}) = \mathcal{Q}(\mathcal{R}^{max})$*.*

Proof. A length multiplicative function $q : W \to \mathbb{R}_+$ is determined by its restriction to the set of simple affine roots and this restriction is a function which is constant on the intersection of the W-orbits of affine roots intersected with the simple affine roots. Conversely every such function on the simple affine roots can be extended uniquely to a length multiplicative function on W. The group $\Omega \simeq X/Q(R_0) \subset W$ of elements of length 0 acts on the set of simple affine roots by diagram automorphisms which preserve the components of the affine Dynkin diagram of the affine root system $R^a = R_0^\vee \times \mathbb{Z}$. The action of Ω on the i-th component factors through the action of $\Omega^{(i)} := X^{(i)}/Q(R_0^{(i)})$. This proves (i). We also see by this that length multiplicative function $q \in \mathcal{Q}(\mathcal{R})$ extends uniquely to a length multiplicative function

for $W(\mathcal{R}^{max})$, since $\alpha^\vee \notin 2Y$ for all $\alpha \in R_0'$ with $\mathcal{R}'$ an indecomposable summand which is not isomorphic to an irreducible root datum of type $C_n^{(1)}$. This proves (ii). $\square$

Given a root datum $\mathcal{R}$ and positive parameter function $q \in \mathcal{Q}(\mathcal{R})$ we define the Macdonald c-function of the pair $(\mathcal{R}, q)$. This is the rational function c on the torus $T = \mathrm{Hom}(X, \mathbb{C})^\times$ defined by

$$c = \prod_{\alpha \in R_{1,+}} c_\alpha, \tag{32}$$

where c_α is defined for $\alpha \in R_1$ by

$$c_\alpha(t,q) := \frac{(1 + q_{\alpha^\vee}^{-1}\alpha(t)^{-1/2})(1 - q_{\alpha^\vee}^{-1}q_{2\alpha^\vee}^{-2}\alpha(t)^{-1/2})}{1 - \alpha(t)^{-1}} \tag{33}$$

Observe that the function c_α is rational in t despite the appearance of the square root $\alpha(t)^{1/2}$. Indeed, if $\alpha/2 \notin X$ then we have $q_{2\alpha^\vee} = 1$, and the numerator simplifies to $(1 - q_{\alpha^\vee}^{-2}\alpha(t)^{-1})$.

The pole order at $t = r \in T$ of the rational function

$$\eta(t) := (c(t)c(t^{-1}))^{-1} \tag{34}$$

is defined as follows. By definition $\eta(t)$ is a product of rational functions of the form $\eta_\alpha := (c_\alpha(t)c_\alpha(t^{-1}))^{-1}$ where α runs over the set $R_{1,+}$. Let $\beta \in R_0$ be the unique root such that α is a positive multiple of β. Then η_α is the pull back via β of a rational function ρ_α on $\mathbb{C}^\times$; we define the pole order of η_α at r to be equal to minus the order of ρ_α at $\beta(r) \in \mathbb{C}^\times$. The pole order $i_{\{r\}}$ of η at $r \in T$ is defined as the sum of these pole orders.

Theorem 2.38. [O3, Theorem 6.1] *For any point $r \in T$, the pole order $i_{\{r\}}$ of $\eta(t)$ at $t = r$ is at most equal to the rank $\mathrm{rk}(R_0)$ of R_0.*

Definition 2.39. *We call $r \in T$ a* residual point *of the pair $(\mathcal{R}, q)$ if $i_{\{r\}} = \mathrm{rk}(X)$. The set of $(\mathcal{R}, q)$-residual points is denoted by $\mathrm{Res}(\mathcal{R}, q)$.*

In particular the set $\mathrm{Res}(\mathcal{R}, q)$ is nonempty only if $\mathcal{R}$ is a semisimple root datum.

The next result is trivial but it explains in conjunction with Proposition 2.37 how residual points for $\mathcal{R}$ can be expressed in terms of residual points of the simple factors of $\mathcal{R}^{max}$:

Lemma 2.40. *Let $\mathcal{R} = (X, R_0, Y, R_0^\vee)$ be a semisimple root datum.*

(i) *Suppose that $\mathcal{R} \to \mathcal{R}'$ is an isogeny which yields an identification $\mathcal{Q} = \mathcal{Q}'$ (e.g. $\mathcal{R}' = \mathcal{R}^{max}$ as in Proposition 2.37). For all $q \in \mathcal{Q}$ we have: $r' \in \mathrm{Res}(\mathcal{R}', q)$ iff $r = r'|_X \in \mathrm{Res}(\mathcal{R}, q)$.*

(ii) *Suppose that $\mathcal{R} = \mathcal{R}^{(1)} \times \cdots \times \mathcal{R}^{(m)}$ is a direct product of simple factors (e.g. if $\mathcal{R} = \mathcal{R}^{max}$ as in Proposition 2.37). Let $T = T^{(1)} \times \cdots \times T^{(m)}$ be the corresponding factorization of T and let $\mathcal{Q}(\mathcal{R}) = \mathcal{Q}^{(1)} \times \cdots \times \mathcal{Q}^{(m)}$ be the corresponding factorization of $\mathcal{Q}$.*

For all $q = (q^{(1)}, \ldots, q^{(m)}) \in \mathcal{Q}$ we have a natural bijection

$$\mathrm{Res}(\mathcal{R}, q) \xrightarrow{\approx} \mathrm{Res}(\mathcal{R}^{(1)}, q^{(1)}) \times \cdots \times \mathrm{Res}(\mathcal{R}^{(m)}, q^{(m)}) \tag{35}$$

such that $r \to (r^{(1)}, \ldots, r^{(m)})$ iff $r = r^{(1)} \ldots r^{(m)}$ with $r^{(i)} \in T^{(i)}$ for all $i = 1, \ldots, m$.

The following result is straightforward as well:

Lemma 2.41. *Let $\mathcal{R}$ be a semisimple root datum with root parameter function $q \in \mathcal{Q}$. Let $r \in T$ with polar decomposition of the form $r = sc$. Let $\mathcal{R}_s = (X, R_{s,0}, Y, R_{s,0}^\vee)$ be the root datum with the root parameters q_s as in Definition 2.5. Then r is a $(\mathcal{R}, q)$-residual point iff r is a $(\mathcal{R}_s, q_s)$-residual point. In particular $\mathcal{R}_s$ is semisimple in this case.*

Let $L \subset T$ be a coset of a subtorus $T^L \subset T$. We decompose the product (34) as follows

$$\eta = \eta_L \eta^L \tag{36}$$

where η_L is the product of the factors c_α where $\alpha \in R_{L,1} \subset R_1$, the subset of R_1 consisting of the roots that are constant on L, and η^L is the product over the remaining roots. We define the order i_L of η at L as the order of η_L at L, viewed as a point of the quotient torus T/T^L. Hence by Theorem 2.38 we have $i_L \leq \mathrm{rank}(R_L)$ for all cosets L, and we define

Definition 2.42. *We call a coset $L \subset T$ a residual coset if $i_L = \mathrm{codim}(L)$ (in particular, $L = T$ is residual). If we denote $L = rT^L$ where $r \in T_L$, the subtorus such that $\mathrm{Lie}(T_L)$ is the orthogonal complement of $\mathrm{Lie}(T^L)$, then L is residual iff r is a residual point for the restriction of η_L to T_L. We define the tempered part of L to be $L^{\mathrm{temp}} := rT^L_u$ (this is well defined).*

Recall the following useful results for residual cosets.

Proposition 2.43. [O3, Lemma 4.1] *Let L be a residual coset, $L \neq T$. Then there exists a residual coset $M \supset L$ such that $\dim(M) = \mathcal{L} + 1$.*

From this result one proves easily by induction to the rank of R_0 (alternatively, it follows from Corollary 2.16 in view of Theorem 2.47):

Theorem 2.44. [O3, compare Theorem 1.1] *The set of residual points is finite.*

We will also need the following results:

Theorem 2.45. [O3, Theorem 7.4] *Define $t^* := \overline{t}^{-1}$. Then $W_0(L^{\mathrm{temp}})^* = W_0 L^{\mathrm{temp}}$.*

Theorem 2.46. [O3, Theorem 6.1] *If $L \neq M$ are residual cosets of T then $L^{\mathrm{temp}} \not\subset M^{\mathrm{temp}}$. Equivalently, the restriction of η^L to L^{temp} is smooth.*

The relevance of the notion of residual cosets stems from:

Theorem 2.47. [O1, Theorem 3.29], [O3, Theorem 6.1] *An orbit $W_0 r \in W_0 \backslash T$ is the central character of a discrete series character of $\mathcal{H}(\mathcal{R}, q)$ iff r is a residual point, and $W_0 r$ is the central character of a tempered character of $\mathcal{H}(\mathcal{R}, q)$ iff $r \in S$, where*

$$S = S(q) = \bigcup_{L \ \text{tempered}} L^{\mathrm{temp}} \tag{37}$$

Remark 2.48. *As we have seen above, $\mathrm{Res}(\mathcal{R}, q) \neq \emptyset$ only if $\mathcal{R}$ is semisimple. By Lemma 2.40 their classification reduces to the case of simple root data. The residual points for simple root data have been classified (*[HO1, Section 4] *and* [O1, Appendix A]*), and various of the above properties of residual points and cosets were first proven by classification. In* [O3] *most of these properties were proved conceptually (with exception of* [O1, Theorem A.14(iii), Theorem A.18]*). In this paper we will only use properties of residual points for which we know a classification-free proof* unless stated otherwise.

2.4. **Generic residual points.** We will study the deformation of discrete series characters with respect to the parameter $q \in \mathcal{Q}$. We begin by studying the dependence of the central characters on $\mathcal{Q}$. We denote the set of all positive real parameter functions for $\mathcal{R}$ by $\mathcal{Q} = \mathcal{Q}(\mathcal{R})$. Recall the following terminology:

Remark 2.49. *We choose a base $\mathbf{q} > 1$ and define $f_s \in \mathbb{R}$ such that $q(s) = \mathbf{q}^{f_s}$ for all $s \in S^{\mathrm{aff}}$. We equip $\mathcal{Q}$ in the obvious way with the structure of the vector group $\mathbb{R}_+^N$ where N denotes the number of W-conjugacy classes in S^{aff}. Given a base $\mathbf{q} > 1$ we identify $\mathcal{Q}$ with the finite dimensional real vector space of real functions $s \to f_s$ on S^{aff} which are constant on W-conjugacy classes. In this sense we speak of (linear)* hyperplanes *in $\mathcal{Q}$ (this notion is independent of $\mathbf{q}$). By a* half line *in $\mathcal{Q}$ we mean a family of parameter functions $q \in \mathcal{Q}$ in which the $f_s \in \mathbb{R}$ are kept fixed and are not all equal to 0 and $\mathbf{q}$ is varying in $\mathbb{R}_{>1}$.*

As was remarked in [O2], it follows easily from [O1, Theorem A.7] that the residual points arise in generic $\mathcal{Q}$-families. Let us state and prove this result precisely.

Definition 2.50. *A real analytic function $r : \mathcal{Q} \to T$ is called a* generic residual point *of $\mathcal{R}$ if there exists an open, dense subset $U \subset \mathcal{Q}$ such that the element $r(q) \in \mathrm{Res}(\mathcal{R}, q)$ for all $q \in U$. The set of generic residual points of $\mathcal{R}$ is denoted by $\mathrm{Res}(\mathcal{R})$.*

Definition 2.51. *Let $r \in \mathrm{Res}(\mathcal{R})$. We call $q \in \mathcal{Q}$ an r-regular (or W_0r-regular) parameter if $r(q) \in \mathrm{Res}(\mathcal{R}, q)$. We denote by $\mathcal{Q}^{reg}_{W_0r} \subset \mathcal{Q}$ the subset of W_0r-regular parameters.*

It is clear that $\mathcal{Q}^{reg}_{W_0r} \subset \mathcal{Q}$ is the complement of a closed real analytic subset (for a more precise statement, see Theorem 2.60). This implies the following basic finiteness result:

Proposition 2.52. *The set $\mathrm{Res}(\mathcal{R})$ of generic residual points is finite and W_0-invariant. This set is nonempty iff $\mathcal{R}$ is semisimple.*

Proof. Suppose that there exist infinitely many distinct generic residual $\mathcal{Q}$-families $q \to r(q)$. Choose countably infinitely many distinct residual families $r_1, r_2, \ldots$. By Baire's theorem we can choose $q \in \mathcal{Q}$ such that the $r_i(q)$ are all residual and mutually distinct. But by Theorem 2.44 there are at most finitely many residual points for q, a contradiction. Hence the set Res is finite. The W_0-invariance is clear. By Theorem 2.38 it follows that this set is empty if the rank of R_0 is not equal to the rank of X. The converse is also clear, because we can easily write down a generic residual point if $\mathcal{R}$ is semisimple. $\square$

2.4.1. *Results on the reduction to simple root systems.* The following result is useful to reduce statements about generic residual points to the case of simple root data.

Lemma 2.53. (i) *Let $\mathcal{R}, \mathcal{R}'$ be as in Lemma 2.40(i). The restriction map $r' \to r = r'|_{\mathcal{Q}\times X}$ is a surjection $\mathrm{Res}(\mathcal{R}') \to \mathrm{Res}(\mathcal{R})$ with fibers of order $|X' : X|$.*

(ii) *Let $\mathcal{R}$ be as in Lemma 2.40(ii). Then we have a natural bijection*

$$\mathrm{Res}(\mathcal{R}) \xrightarrow{\approx} \mathrm{Res}(\mathcal{R}^{(1)}) \times \cdots \times \mathrm{Res}(\mathcal{R}^{(m)}) \tag{38}$$

such that $r \to (r^{(1)}, \ldots, r^{(m)})$ iff $r(q^{(1)}, \ldots, q^{(m)}) = r^{(1)}(q^{(1)}) \ldots r^{(m)}(q^{(m)})$ with $r^{(i)}(q^{(i)}) \in T^{(i)}$ for all $i = 1, \ldots, m$ and all $q = (q^{(1)}, \ldots, q^{(m)}) \in \mathcal{Q}$.

(iii) *Let $\mathcal{R}$ be arbitrary semisimple and let $\mathcal{Q} = \mathcal{Q}^{(1)} \times \cdots \times \mathcal{Q}^{(m)}$ be the decomposition of $\mathcal{Q} = \mathcal{Q}(\mathcal{R})$ as in Proposition 2.37(i). Suppose that $\mathcal{Q}' \subset \mathcal{Q}$ is a connected closed subgroup of $\mathcal{Q}$ such that for each $i = 1, \dots, m$ the projection $\pi_i : \mathcal{Q}' \to \mathcal{Q}^{(i)}$ is surjective. Let $r' : \mathcal{Q}' \to T$ be real analytic with the property that $r'(q') \in \mathrm{Res}(\mathcal{R}, q')$ for almost all $q' \in \mathcal{Q}'$. Then there exists a unique $r \in \mathrm{Res}(\mathcal{R})$ such that $r' = r|_{\mathcal{Q}'}$.*

Proof. The first two assertions are clear so let us look at (iii). Let $\tilde{r'} : \mathcal{Q}' \to T^{max} = T^{(1)} \times \cdots \times T^{(m)}$ be a lifting of r'. Choose homomorphisms $\phi_i : \mathcal{Q}^{(i)} \to \mathcal{Q}'$ such that $\pi_i \circ \phi_i = \mathrm{id}_{\mathcal{Q}^{(i)}}$ for all i. Lemma 2.40 implies that the map $\tilde{r}_i : \mathcal{Q}^{(i)} \to T^{(i)}$ defined by $\tilde{r}^{(i)}(q^{(i)}) := (\tilde{r'}(\phi_i(q^{(i)})))^{(i)}$ is a generic residual point for $\mathcal{R}^{(i)}$. Let $\tilde{r} \in \mathrm{Res}(\mathcal{R})$ correspond to $(\tilde{r}^{(1)}, \dots, \tilde{r}^{(m)})$ (using the notation of (ii)). Then (i) implies that $r = \tilde{r}|_{\mathcal{Q} \times X}$ meets the requirement. If r_1 also meets the requirement let $\widetilde{r_1}$ be the unique lifting of r_1 to $\mathrm{Res}(\mathcal{R}^{max})$ such that $\widetilde{r_1}|_{\mathcal{Q}'} = \tilde{r'}$. Then it is clear that for all i we must have $\widetilde{r_1}^{(i)} = \tilde{r}^{(i)}$. The uniqueness follows. □

Recall the result of Lemma 2.41. We see that if $r = sc$ is a residual point then $s \in T_u$ belongs to the finite set of points with the property that $\mathcal{R}_s$ is semisimple. In particular, if $r : \mathcal{Q} \to T$ is a generic residual point then the unitary part s of r is independent of $q \in \mathcal{Q}$ and $\mathcal{R}_s$ is semisimple.

Corollary 2.54. *Suppose that $\mathcal{R}$ is semisimple and $s \in T_u$ is such that $\mathcal{R}_s$ is semisimple. Let $\phi_s : \mathcal{Q}(\mathcal{R}) \to \mathcal{Q}(\mathcal{R}_s)$ denote the homomorphism $q \to q_s$.*

(i) *Let $\mathrm{Res}^s(\mathcal{R})$ denote the set of generic residual points r with unitary part s. There exists a natural bijection*

$$\begin{aligned} \Phi_s : \mathrm{Res}^s(\mathcal{R}) &\to \mathrm{Res}^s(\mathcal{R}_s) \\ r &\to r \circ \phi_s \end{aligned}$$

(ii) *Using the notation of Definition 2.5, we have a natural bijection*

$$\begin{aligned} \Phi^{W_0}_{W_0 s} : W_0 \backslash \mathrm{Res}^{W_0 s}(\mathcal{R}) &\to \Gamma_s \backslash (W(R_{s,1}) \backslash \mathrm{Res}^s(\mathcal{R}_s)) \\ W_0 r &\to \Gamma_s W(R_{s,1})(r \circ \phi_s) \end{aligned}$$

Here $W_0 \backslash \mathrm{Res}^{W_0 s}(\mathcal{R})$ denotes the set of W_0-orbits of generic residual points whose unitary part is $W_0 s$.

Proof. The image $\mathcal{Q}' = \phi(\mathcal{Q}) \subset \mathcal{Q}_s$ satisfies the condition as in Lemma 2.53(iii). The result (i) then follows from Lemma 2.41 and Lemma 2.53(iii). The assertion (ii) follows from (i) and Definition 2.5. □

The previous Corollary reduces the classification of the set $\mathrm{Res}(\mathcal{R})$ to the classification of those elements $r \in \mathrm{Res}(\mathcal{R})$ which are of the form $r = sc$ where s is W_0-invariant. In this case we further reduce to the level of the degenerate Hecke algebra:

Definition 2.55. *Let $R_1 \subset V^*$ be a semisimple, reduced root system and let $\mathcal{K}$ be the space of W_0-invariant real valued functions on R_1. We denote by $\mathrm{Res}^{lin}(R_1)$ the set of linear maps $\xi : \mathcal{K} \to V$ such that for almost all k the point $\xi(k) \in V$ is (R_1, k)-residual in the sense of* [HO1]*, i.e.*

$$|\{\alpha \in R_1 \mid \alpha(\xi(k)) = k_\alpha\}| = |\{\alpha \in R_1 \mid \alpha(\xi(k)) = 0\}| + \dim(V) \tag{39}$$

We refer to this set as the set of generic linear residual points associated to the root system R_1.

Proposition 2.56. *Let $\mathcal{R}$ be semisimple and let $s \in T_u$ be W_0-invariant. Let $\mathcal{K}$ be the vector space of real W_0-invariant functions on R_1, and given $q \in \mathcal{Q}$ let $k_s \in \mathcal{K}$ be the W_0-invariant function on R_1 associated to q by the formulas of equation (26). Let $r = sc$ be a generic $\mathcal{R}$-residual point.*

(i) *There exists a unique generic linear residual point $\xi \in \mathrm{Res}^{lin}(R_1)$ such that $\alpha(c(q)) = e^{\alpha(\xi(k_s))}$ for all $\alpha \in R_1$ and all $q \in \mathcal{Q}$ (where k_s is related to q as above). We express this relation between r and ξ by $r = s\exp(\xi)$.*

(ii) *This yields a W_0-equivariant bijection between $\mathrm{Res}^{W_0 s}(\mathcal{R})$ and $\mathrm{Res}^{lin}(R_1)$.*

(iii) *For all $q \in \mathcal{Q}$ we have: $r(q)$ is $(\mathcal{R}, q)$-residual iff $\xi(k_s)$ is (R_1, k_s)-residual (in the sense of* [HO1]*).*

(iv) *The generic linear residual points ξ are rational in the sense that the $\alpha(\xi(k))$ is a rational linear combination of the values k_β for all $\alpha \in R_1$.*

Proof. The existence of ξ is a special case of [O1, Theorem A.7], and the uniqueness is clear since R_1 spans V^*. Similarly (ii) follows from [O1, Theorem A.7]. The rationality of ξ follows from the fact that the set of roots contributing to the pole order of c at r span a sublattice of X of finite index as a consequence of Theorem 2.38. $\square$

The following reduction to simple root systems follows easily from the definitions:

Proposition 2.57. *Let $R_1 = R_{1,1}, \dots, R_{N,1}$ be the decomposition in simple root systems. Then $\mathcal{K} = \mathcal{K}_1 \times \cdots \times \mathcal{K}_N$ and $\mathrm{Res}^{lin}(R_1) = \mathrm{Res}^{lin}(R_{1,1}) \times \cdots \times \mathrm{Res}^{lin}(R_{N,1})$.*

2.4.2. *Rationality results for generic residual points.* Nothing that follows in this paper depends on the results in this paragraph in any essential way, but these results simplify notations and reveal certain basic facts. The proofs in this paragraph depend on the classification of positive generic residual points for irreducible root systems.

Theorem 2.58. *Let $\mathcal{R}$ be a semisimple root datum, and let $r : \mathcal{Q} \to T$ be a generic residual point of the form $r = sc$. For all $x \in X$ the expression $x(c) \in \Lambda$ is a monomial in the generators $v(s)^{\pm 1}$ with $s \in S$. Here $v(s)$ is viewed as a function on $\mathcal{Q}$ by $(v(s))(q) := q(v(s))$. In other words, r is (the restriction to $\mathcal{Q}$ of) a $\mathcal{Q}_c$-valued point of T.*

Proof. Using Lemma 2.53 if suffices to show this for $\mathcal{R} = (X, R_0, Y, R_0^\vee, F_0)$ with R_0 irreducible and X the weight lattice of R_1. By Corollary 2.54 is suffices to consider the case where $s \in T$ is W_0-invariant. Then we are in the situation of Proposition 2.56. In terms of the rational linear function $\xi : \mathcal{K} \to V$ of Proposition 2.56 the assertion amounts to showing that 2ξ is integral, i.e. $x(2\xi)$ is an integral linear combination of the functions k_β (with $\beta \in R_1$) for all integral weights x. We call ξ a generic residual point for R_1 (in the sense of [HO1]).

If $R_1 = A_n$ it is easy to see that 2ξ is integral (even even for even n).

If $R_1 = B_n$ it suffices to remark that the integrality of ξ with respect to the root lattice follows from the description of the residual points as in [HO1, Section 4] (also see Section 6).

The generic residual points for R_1 of type C_n are in bijection to those of type B_n as follows. Let k_1 denote the parameter of the C_n roots of the form $\pm e_i \pm e_j$

and k_2 the parameter of the C_n roots $\pm 2e_i$. If ξ' is a generic B_n-residual point then $\xi(k_1, k_2) = \xi'(k_1, k_2/2)$ is a generic C_n residual point. This sets up a bijective correspondence between the generic residual points of B_n and of C_n. Hence if ξ is residual for C_n then 2ξ is integral with respect to the root lattice of B_n, which is equal to the the weight lattice of C_n.

If R_1 is of type D_n or E_n we use that ξ is integral with respect to the root lattice [O1, Corollary B2]. In order to check the integrality of 2ξ with respect to the weight lattice one needs to check in addition the integrality of $x(2\xi)$ with respect to the minuscule fundamental weights. This is an easy verification using the explicit descriptions of the Bala-Carter diagrams of the distinguished parabolic subgroups in [Car, Section 5.9] (see [O1, Appendix B] for the explanation of the relation between residual points and Bala-Carter diagrams for the simply laced types) and the table 1 of [Hum1, Chapter III, Section 13.2] expressing the fundamental weights in the simple roots. For $R_1 = D_n$ there are 3 minuscule fundamental weights to check, and for $R_1 = E_6$ there are 2 of these. For E_7 and E_8 the integrality of ξ with respect to the root lattice suffices since the index of the root lattice in the weight lattice is at most 2.

For F_4 and G_2 the root lattice is equal to the weight lattice. In these cases the result follows simply from the tables in [HO1, Section 4]. $\square$

We introduce the following notation

Definition 2.59. *Let $r = sc \in \mathrm{Res}(\mathcal{R})$. Recall that for all $\alpha \in R_1$ we have $\alpha(r) = \alpha(s)\alpha(c)$ with $\alpha(s)$ a root of 1 and $\alpha(c)$ a monomial in the variables $v_{\beta^\vee}^{\pm 1}$ (with $\beta^\vee \in R_{\mathrm{nr}}$) as described above. Define*

$$\begin{aligned} R_{r,1}^{p,-} &= \{\alpha \in R_0 \cap R_1 \mid v_{\alpha^\vee}^2 \alpha(r) - 1 = 0\} \cup \{2\beta \in R_1 \backslash R_0 \mid v_{\beta^\vee/2} v_{\beta^\vee}^2 \beta(r) - 1 = 0\} \\ R_{r,1}^{p,+} &= \{2\beta \in R_1 \backslash R_0 \mid v_{\beta^\vee/2} \beta(r) + 1 = 0\} \\ R_{r,1}^{z} &= \{\alpha \in R_1 \mid \alpha(r) - 1 = 0\} \end{aligned}$$

and we define an element $m_{W_0 r} \in K(\Lambda)$ in the quotient field $K(\Lambda)$ of Λ by

$$m_{W_0 r} := \frac{\prod_{\alpha \in R_1 \backslash R_{r,1}^z} (\alpha(r)^{-1} - 1)}{\prod_{\alpha \in R_1 \backslash R_{r,1}^{p,+}} (v_{\alpha^\vee}^{-1} \alpha(r)^{-1/2} + 1) \prod_{\alpha \in R_1 \backslash R_{r,1}^{p,-}} (v_{\alpha^\vee}^{-1} v_{2\alpha^\vee}^{-2} \alpha(r)^{-1/2} - 1)} \tag{40}$$

As before, if $\alpha \in R_0 \cap R_1$ then $v_{2\alpha^\vee} = 1$ and the corresponding terms in the denominator simplify to $(v_{\alpha^\vee}^{-2}\alpha(r)^{-1} - 1)$. Therefore the expression is rational in the values $\alpha(r)$ with $\alpha \in R_0$. Observe that the above definition of $m_{W_0 r}$ is indeed independent of the choice of r in the W_0-orbit $W_0 r$, justifying the notation $m_{W_0 r}$.

Theorem 2.60. *Let r be a generic residual point. We view the generators $v(s)$ of Λ as functions on $\mathcal{Q}$ via $v(s)(q) := q(v(s))$ as before. The function $m_{W_0 r}$ is real analytic on $\mathcal{Q}$. The set of r-regular points $\mathcal{Q}_{W_0 r}^{\mathrm{reg}} := \{q \in \mathcal{Q} \mid r(q) \in \mathrm{Res}(\mathcal{R}, q)\}$ of $\mathcal{Q}$ is the complement of the zero locus of $m_{W_0 r}$ in $\mathcal{Q}$. In particular this set is the complement of a union of finitely many (rational) hyperplanes in $\mathcal{Q}$.*

Proof. Since $r(q)$ is generically residual it is clear that $|R_{r,1}^{p,+} \cup R_{r,1}^{p,-}| - |R_{r,1}^z| = \mathrm{rank}(X)$. By Theorem 2.38 it is therefore clear that for all $q \in \mathcal{Q}$ the number of factors that are zero at q in the numerator of $m_{W_0 r}$ has to be at least equal to the

number of factors that are zero at q in the denominator. This implies that $m_{W_0 r}$ is real analytic on $\mathcal{Q}$, and that the zero locus of $m_{W_0 r}$ in $\mathcal{Q}$ is precisely the set of q such that $r(q)$ is not residual. $\square$

Definition 2.61. *Let $q \in \mathcal{Q}$. We define $\mathrm{Res}_q(\mathcal{R}) = \{r \in \Delta(\mathcal{R}) \mid r(q) \in \mathrm{Res}(\mathcal{R}, q)\}$. Or $\mathrm{Res}_q(\mathcal{R})$ is the set of generic residual points whose specialization at q is residual.*

Let $r = sc \in \mathrm{Res}(\mathcal{R})$. By Lemma 2.41 the evaluations $x(s)$ with $x \in X$ are roots of unity. Let $K \supset \mathbb{Q}$ be the Galois extension of $\mathbb{Q}$ generated by the values $x(s)$ with $x \in X$. Theorem 2.58 implies that for all $x \in X$ we have $x(\tilde{r}) \in K[v(s)^{\pm 1} : s \in S]$, the ring of Laurent polynomials in the variables $v(s)^{\pm 1}$ (with $s \in S$) with coefficients in K. Let $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$. By the above there is a canonical action $r \to \sigma(r)$ of $\mathrm{Gal}(K/\mathbb{Q})$ on $\mathrm{Res}(\mathcal{R})$ characterized by $x(\widetilde{\sigma(r)}) = \sigma(x(\tilde{r}))$ for all $x \in X$, where σ on the right hand side is acting on the coefficients of $x(\tilde{r}) \in \Lambda$ (these are indeed elements of Λ with algebraic coefficients, by Lemma 2.41 and Theorem 2.58).

Proposition 2.62. *Let $\mathcal{R}$ be a semisimple root datum.*

(i) *Let $r \in \mathrm{Res}(\mathcal{R})$ and $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$. Then $\sigma(r)|_{Q(R_0)} \in W_0 r|_{Q(R_0)}$ where $Q(R_0) \subset X$ denotes the root lattice of R_0.*

(ii) *For all $r \in \mathrm{Res}(\mathcal{R})$ we have $m_{W_0 r} \in K(\Lambda_{\mathbb{Z}})$, the quotient field of the subring $\Lambda_{\mathbb{Z}} := \mathbb{Z}[v([s])^{\pm 1} : [s] \in \tilde{S}] \subset \Lambda$ of Λ.*

(iii) *In the situation of Lemma 2.53(i) we have $m_{W_0 r} = m_{W_0 r'}$ and in the situation of Lemma 2.53(ii) we have $m_{W_0 r}(q) = m_{W_0^{(1)} r^{(1)}}(q^{(1)}) \dots m_{W_0^{(k)} r^{(k)}}(q^{(k)})$.*

Proof. The first assertion follows from the proof of [O1, Proposition 3.27]. Then (ii) follows from (i) by the fact that $m_{W_0 r}$ only depends on the restriction of r to $Q(R_0)$ and the fact that the assignment $r \to m_{W_0 r}$ is W_0-invariant. The assertions of (iii) are trivial. $\square$

2.4.3. *Deformation of residual points in the parameter $\mathcal{Q}$.* The following result is very important: it says that all residual points are obtained from specialization of the generic residual points.

Proposition 2.63. *Let $\mathcal{R}$ be a semisimple based root datum. The evaluation map $\mathrm{ev}_q : \mathrm{Res}_q(\mathcal{R}) \to \mathrm{Res}(\mathcal{R}, q)$ given by $\mathrm{ev}_q(r) = r(q)$ is surjective for all $q \in \mathcal{Q}$.*

Proof. We prove this fact by induction to the rank of R_0. If the rank of R_0 is one the assertion can be verified by an easy inspection. Assume that the result holds for all maximal proper parabolic subsystems of R_0. Let $r_0 \in T$ be a residual point for the parameter value $q_0 \in \mathcal{Q}$. By Proposition 2.43 we know that that there exists a residual line $L_0 = r_{L,0} T^L$ where $r_{L,0} \in T_L$ is a residual point for a proper maximal parabolic subsystem $R_L \subset R_0$ with the property that $r_0 \in L_0$. By the induction hypothesis, $L_0 = L(q_0)$ for a generic family of residual lines $L(q) = r_L(q) T^L$ (in other words, the $\mathcal{R}_L$-residual point $r_{L,0}$ is the specialization $r_{L,0} = r_L(q_0)$ at q_0 of a generic $\mathcal{R}_L$ residual point r_L). By Theorem 2.38 and Definition 2.42 it follows easily that for each fixed $q \in \mathcal{Q}$ such that $r_L(q)$ is residual the rational function η^L (see (36)) on $L(q)$ has poles of order at most one on $L(q)$, and $x \in L(q)$ is $(\mathcal{R}, q)$-residual if and only if x is a pole of $\eta^L(\cdot, q)$. In particular r_0 is a simple pole of $\eta^L(\cdot, q_0)$. Considering the form of the factors in the denominator of η^L this implies easily that r_0 is the specialization at $q = q_0$ of least one Q-family of the form $q \to r(q) \in L(q)$ such that $r(q)$ is residual for all q in an open neighborhood of q_0. Hence $r \in \mathrm{Res}_q(\mathcal{R})$ and $\mathrm{ev}_{q_0}(r) = r(q_0) = r_0$ as desired. $\square$

Definition 2.64. *Let $\mathcal{R}$ be a semisimple root datum and let $r \in \mathrm{Res}(\mathcal{R})$. We call $q \in \mathcal{Q}^{\mathrm{reg}}_{W_0 r}$ an r-*generic *(or $W_0 r$-generic) parameter if for all $r' \in \mathrm{Res}(\mathcal{R})$ the equality $W_0 r'(q) = W_0 r(q)$ implies that $r' \in W_0 r$. The set of r-generic parameters is denoted by $\mathcal{Q}^{\mathrm{gen}}_{W_0 r}$. We define the set $\mathcal{Q}^{\mathrm{gen}}$ of* generic *parameters by $\mathcal{Q}^{\mathrm{gen}} = \cap_{r \in \mathrm{Res}(\mathcal{R})} \mathcal{Q}^{\mathrm{gen}}_{W_0 r}$.*

Proposition 2.65. *Let $\mathcal{R}$ be a semisimple root datum. For all $r \in \mathrm{Res}(\mathcal{R})$ the set $\mathcal{Q}^{\mathrm{gen}}_{W_0 r}$ is the complement of a finite collection of rational hyperplanes in $\mathcal{Q}$.*

Proof. This follows easily from Corollary 2.54 and Proposition 2.56. □

The proof of the following important Proposition depends on the classification of residual points.

Proposition 2.66. *Recall that the central support of the set of tempered irreducible characters of $\mathcal{H}(\mathcal{R}, q)$ is given by the union $S(q) = \cup_L L^{\mathrm{temp}}$ (union over the set $(\mathcal{R}, q)$-residual cosets $L \subset T$) (see Theorem 2.47). Let $S_i(q) = \cup_L L^{\mathrm{temp}} \subset S(q)$ denote the subset of $S(q)$ where the union is taken only over the residual cosets of dimension at least i. The sets $\cup_{q \in \mathcal{Q}}(q, S_i(q)) \subset \mathcal{Q} \times T$ are closed for all i.*

Proof. In view of Definition 2.42 it is clear that it suffices to show that if $r \in \mathrm{Res}(\mathcal{R})$ and $q_0 \in \mathcal{Q}^{\mathrm{sing}}_{W_0 r}$, then there exists a residual coset L such that $r(q_0) \in L^{\mathrm{temp}}$. By [O1, Theorem A.7] this reduces to the statement that if c is a positive generic residual point, then $c(q_0)$ coincides with the center of a positive residual coset. Since the collection of centers of *positive* residual cosets does not depend on the choice of the lattice X we may replace X by X^{max} (as in Proposition 2.37). Since $\mathcal{R}^{max}$ is a direct sum of irreducible summands this shows that it suffices to prove the statement for a root datum $\mathcal{R}$ with R_0 irreducible.

In the case where R_0 is simply laced this follows from the remark that $\mathcal{Q}^{\mathrm{sing}}_{W_0 r} = \{q_0 = 1\}$ for all $r \in \mathrm{Res}(\mathcal{R})$. By Lemma 2.41 we have $r(1) = e$, which is the center of $T^{\mathrm{temp}} = T_u$. If R_0 is of type B_n or C_n, then this is [Slo2, Proposition 4.15]. For type G_2 and F_4 it can be read off from the tables [HO1, Table 4.10, Table 4.15]. □

3. Continuous families of discrete series

In this section we show that every discrete series character of $\mathcal{H} = \mathcal{H}(\mathcal{R}, q)$ is the specialization of a unique maximal "continuous parameter family" of discrete series characters. Using this fact and our results on $EP_{\mathcal{H}}$ the discrete series can be parametrized explicitly for all irreducible root data $\mathcal{R}$ which are not simply laced. An important ingredient is the fact that the central characters of the irreducible discrete series characters are precisely the W_0-orbits of residual points.

Another main result in this section states that the formal degree of a continuous family of irreducible discrete series characters is a rational function on $\mathcal{Q}$ with rational coefficients. This function has a product expansion in terms of the central character of the family, and an alternating sum expansion in terms of the branching multiplicities of the discrete series representation to finite dimensional Hecke subalgebras.

3.1. **Parameter deformation of the discrete series.**

In this subsection we show that each irreducible discrete series character is a specialization in the parameter q of a unique continuous $\mathcal{Q}$-family of irreducible discrete series characters.

It is useful to remark that such deformations are well understood for *scaling deformations* of the parameters along half lines. What we are about to discuss in

this subsection is what happens for general deformations. Therefore this yields no extra information whatsoever for the simply laced cases. On the other hand, for the non-simply laced root systems the method turns out to be sufficient in most cases to distinguish the irreducible discrete series characters with the same central character form each other, and parametrize them by continuous $\mathcal{Q}$-families of discrete series characters.

Definition 3.1. *Let $\mathcal{R}$ be a semisimple root datum, $q_0 \in \mathcal{Q}$, and let $r_0 \in T$ be a $(\mathcal{R}, q_0)$-residual point. We denote by $\mathcal{P}(r_0) = \{W_0 r \in W_0 \backslash \mathrm{Res}(\mathcal{R}) \mid W_0 r(q_0) = W_0 r_0\}$ the finite set of W_0-orbits of generic residual points which coalesce at $W_0 r_0$ for the parameter value $q = q_0$.*

Lemma 3.2. *Let $r_0 = s_0 c_0$ be a $(\mathcal{R}, q_0)$-residual point, and let $0 < \epsilon < 1/3$.*

There exists an open neighborhood $U \subset \mathcal{Q}$ of q_0 and a Hermitian element $z \in \mathbb{C}[T]^{W_0}$ such that

(i) *z is positive on $S(q)$ for all $q \in \mathcal{Q}$.*
(ii) *$z(t) < \epsilon$ for all $q \in U$ and $t \in S(q) \backslash \{W_0 r(q) \mid r \in \mathcal{P}(r_0)\}$.*
(iii) *There exists an $M \geq 1$ such that $1 - \epsilon < z(W_0 r(q)) < M$ for all $q \in U$ and $r \in \mathcal{P}(r_0)$.*

Proof. According to [O1, Lemma 3.5] for any $\delta > 0$ there exist elements $a \in \mathbb{C}[T]^{W_0}$ such that $a(W_0 r_0) = 1$ and such that the uniform norm of a on $S_c(q_0)$ is smaller than δ for all centers c such that $W_0 c \neq W_0 c_0$. By Theorem 2.46 we know that r_0 is disjoint from the union of the tempered residual cosets of dimension at least 1 (in particular, $c_0 \neq e$). Hence we can multiply a by further factors in order to make sure that a is equal to zero on all tempered residual cosets contained in $S_{c_0}(q_0)$ other than r_0. By taking δ small enough we can arrange that the uniform norm of a on all components of $S(q_0)$ other than the points of $W_0 r_0$ is smaller than ϵ. Define $z \in \mathbb{C}[T]^{W_0}$ by $z(t) := a(t)\overline{a(\bar{t}^{-1})}$. Using Theorem 2.45 we see that $z(r_0) = 1$ and that z is nonnegative on $S(q)$ (for all $q \in \mathcal{Q}$). This proves (i).

Define two open subsets $V_+ := \{t \in T \mid |z(t)| > 1 - \epsilon\}$ and $V_- := \{t \in T \mid |z(t)| < \epsilon\}$ of T. By Proposition 2.66 we see that for all $q \in \mathcal{Q}$ the support $S(q)$ is the following union of compact subsets

$$S(q) = \bigcup_{P} \bigcup_{r \in \mathrm{Res}(\mathcal{R}_P)} W_0 r(q) T_u^P \tag{41}$$

Put $W_0 r(q) T_u^P = S(P, r, q)$. By the above it is clear that $S(P, r, q_0) \subset V_+$ iff $R_P = R_0$ and $W_0 r \in \mathcal{P}(r_0)$. On the other hand, $S(P, r, q_0) \subset V_-$ iff $R_P = R_0$ and $W_0 r \notin \mathcal{P}(R_0)$ or if $R_P \neq R_0$. By the compactness of the sets T_u^P and the continuity of the generic residual cosets $r \in \mathrm{Res}(\mathcal{R}_\mathcal{P})$ (viewed as functions on $\mathcal{Q}$) it is clear that there exists an open neighborhood U of q_0 such that for all $q \in U$, and for all pairs (P, r) we have: $S(P, r, q) \subset V_-$ iff $S(P, r, q_0)$ and $S(P, r, q) \in V_+$ iff $S(P, r, q_0) \in V_+$. Hence for all $q \in U$ we have

$$S(q) = (S(q) \cap V_+) \cup (S(q) \cap V_-) \tag{42}$$

and $S(q) \cap V_+ = \mathcal{P}(r_0)(q)$. From this we easily deduce (ii) and (iii). □

Let $L^2(W)$ denote the abstract Hilbert space with Hilbert basis $(\tilde{N}_w)_{w \in W}$ indexed by the elements of W. We identify $L^2(W)$ with the Hilbert completion $L^2(\mathcal{H}(\mathcal{R}, q))$ (for any fixed $q \in \mathcal{Q}$) by identifying $\tilde{N}_w \in L^2(W)$ with the basis element $N_w \in$

$\mathcal{H}(\mathcal{R},q)$. In this way $L^2(W)$ comes equipped with the structure of a module over the C^*-algebra completion of the pre-C^*-algebra $\mathcal{H}(\mathcal{R},q)$. By abuse of notation we will denote the basis elements $\tilde{N}_w$ of the module $L^2(W)$ simply by N_w. Similarly we use the notation $\mathcal{S}(W)$ for the abstract Fréchet space of functions on W which are of rapid decay with respect to the norm function $\mathcal{N}$ on W. For each fixed $q \in \mathcal{Q}$ we identify $\mathcal{S}(W)$ with the Fréchet algebra completion $\mathcal{S}(\mathcal{R},q)$ of $\mathcal{H}(\mathcal{R},q)$.

Given $q \in \mathcal{Q}$ and $z \in \mathbb{C}[T]^{W_0}$ let $z_q \in \mathcal{H}(\mathcal{R},q)$ denote the element z viewed as an element of $\mathcal{H}(\mathcal{R},q)$ via the isomorphism defined by the Bernstein basis of the center $\mathcal{Z}(q)$ of $\mathcal{H}(\mathcal{R},q)$ with $\mathbb{C}[T]^{W_0}$. The above Lemma implies that $z_q \in \mathcal{H}(\mathcal{R},q)$ is a positive central element such that if $q \in U$ its spectrum on $L^2(\mathcal{H}(\mathcal{R},q))$ is contained in $[0,\epsilon) \cup (1-\epsilon, M]$.

Theorem 3.3. *Let U, $M > 0$ and $\epsilon > 0$ be as in the previous Lemma. Let $e_q := p_{>1-\epsilon}(z_q) \in \mathcal{S}(\mathcal{R},q)$ denote the element of $\mathcal{S}(\mathcal{R},q)$ obtained by holomorphic calculus applied to $z_q \in \mathcal{H}(\mathcal{R},q)$ with respect to a function $p_{>1-\epsilon}$ on the spectrum that is equal to 0 in an open neighborhood of $[1,\epsilon]$ and is equal to 1 on an open neighborhood of $[1-\epsilon, M]$.*

(i) *For all $q \in U$, $e_q \in \mathcal{S}(\mathcal{R},q)$ is a self-adjoint, central idempotent.*
(ii) *For all $q \in U$ we have an orthogonal decomposition*

$$e_q = \sum_{W_0 r \in \mathcal{P}(r_0)} \sum_{\delta(q) \in \Delta_{W_0 r(q)}(\mathcal{R},q)} e_{\delta(q),q} \tag{43}$$

where $e_{\delta(q),q}$ is the primitive central idempotent of $\mathcal{S}(\mathcal{R},q)$ corresponding to the irreducible discrete series character $\delta(q) \in \Delta_{W_0 r(q)}(\mathcal{R},q)$ (the set of irreducible discrete series characters of $\mathcal{H}(\mathcal{R},q_0)$ with central character $W_0 r_0$).
(iii) *For all $q \in U$ the two-sided ideal $\mathcal{I}_q := e_q \mathcal{S}(\mathcal{R},q) \subset \mathcal{S}(\mathcal{R},q)$ is a finite dimensional, semisimple, involutive subalgebra of $\mathcal{S}(\mathcal{R},q)$.*
(iv) *The family $q \to e_q \in \mathcal{S}(\mathcal{R},q) \simeq \mathcal{S}(W)$ is continuous with respect to the parameter $q \in U$.*
(v) *The dimension $\dim_{\mathbb{C}}(\mathcal{I}_q)$ is independent of $q \in U$.*
(vi) *The isomorphism class of $\mathcal{I}_q$ viewed as a (finite dimensional) C^*-algebra is independent of $q \in U$.*

Proof. By the previous Lemma it is clear that $p_{>1-\epsilon}$ is holomorphic on the spectrum of z_q, hence we may apply holomorphic functional calculus. Hence (i) follows from the fact that $\mathcal{S}$ is closed for holomorphic functional calculus, see Theorem 2.23, and the basic properties of the holomorphic functional calculus. The assertion (ii) follows from the previous Lemma and the definition of the idempotent e_q. The finite dimensionality of $\mathcal{I}_q$ follows simply from (ii). Clearly $\mathcal{I}_q$ is an involutive algebra because e_q is central and self-adjoint. Thus the trace τ and the anti-involution $*$ give rise to a positive definite Hermitian inner product on $\mathcal{I}_q$ with the property $(ab,c) = (b,a^*c)$. Hence $\mathcal{I}_q$ is a semisimple subalgebra, proving (iii). It is easy to see that $U \ni q \to z_q \in \mathcal{S}(W)$ is a continuous family (by expressing z in the N_w basis of $\mathcal{H}(\mathcal{R},q)$). Hence (iv) follows from the continuity of the holomorphic functional calculus, see Theorem 2.23. For (v) we first remark that it is clear that for all $q \in U$ the projection $\lambda(e_q) \in \mathfrak{B}(L^2(\mathcal{H}(\mathcal{R},q)))$ (where λ denotes the left regular representation) is of finite rank (since only finitely many central characters support the image of e_q by construction). On the other hand it is clear from Theorem 2.23 and [Sol, Proposition 5.6] that this family of projections is norm continuous

in $\mathfrak{B}(L^2(\mathcal{H}(\mathcal{R},q)))$, implying in particular that the rank is constant in the family. Finally observe that $\mathcal{I}_q = \lambda(e_q)(L^2(\mathcal{H}(\mathcal{R},q)))$. In order to prove (vi) we use the notion of approximate matrix units in a C^*-algebra [BKR, Definition 2.2]. Let $m^{(i)}_{j,k}(q_0)$ be a basis of matrix units of $\mathcal{I}_{q_0}$. Given an element $q \in U$ we define $\tilde{m}^{(i)}_{j,k}(q) = e_q \cdot m^{(i)}_{j,k}(q_0)$, where in the right hand side we view $m^{(i)}_{j,k}(q_0)$ as an element of $\mathcal{S}(\mathcal{R},q)$ via the canonical isomorphism $\mathcal{S}(W) \simeq \mathcal{S}(\mathcal{R},q)$. Let $\epsilon' > 0$. By (iv), (v) and [Sol, Proposition 5.6] we obtain that there exists an open neighborhood $q_0 \in U_{\epsilon'} \subset U$ of q_0 such that for all $q \in U_{\epsilon'}$ the elements $\tilde{m}^{(i)}_{j,k}(q)$ form a basis of ϵ'-approximate matrix units of $\mathcal{I}_q$. This means that for all i,j,k,l,m,n and for all $q \in U_{\epsilon'}$ we have

$$\|\tilde{m}^{(i)}_{j,k}(q)\tilde{m}^{(l)}_{m,n}(q) - \delta_{i,l}\delta_{k,m}\tilde{m}^{(i)}_{j,n}(q)\| < \epsilon' \tag{44}$$

and

$$\|\tilde{m}^{(i)}_{j,k} - (\tilde{m}^{(i)}_{k,j})^*\| < \epsilon' \tag{45}$$

(where the norm refers to the C^*-algebra norm). Now [BKR, Lemma 2.3] implies that for $\epsilon' > 0$ sufficiently small there exists a basis of matrix units $m^{(i)}_{j,k}(q)$ of $\mathcal{I}_q$ with the property that for all i,j,k:

$$\|\tilde{m}^{(i)}_{j,k}(q) - m^{(i)}_{j,k}(q)\| < \epsilon' \tag{46}$$

In particular it follows that $\mathcal{I}_q$ for $q \in U_{\epsilon'}$ is isomorphic to $\mathcal{I}_{q_0}$ as a finite dimensional C^*-algebra. Using a suitable open covering of U this result extends easily to $q \in U$, proving (vi). $\square$

Theorem 3.4. *Keep the notations as in Theorem 3.3. Let $r_0 \in \mathrm{Res}(\mathcal{R},q_0)$.*

(i) *There exists an open neighborhood U of q_0 such that for each $\delta_0 \in \Delta_{W_0 r_0}(\mathcal{R},q_0)$ there exists a unique family of primitive central idempotents $U \ni q \to e_{\delta(q),q} \in \mathcal{S}(\mathcal{R},q) = \mathcal{S}(W)$ with the following properties:*
  - (a) *$\delta(q_0) = \delta_0$.*
  - (b) *The function $U \ni q \to \lambda(e_{\delta(q),q}, q) \in \mathcal{B}(L^2(W))$ is continuous.*
  - (c) *For all $q \in U$, the value $e_{\delta(q),q} \in \mathcal{I}_q$ of this function is a primitive central idempotent.*
  - (d) *The degree of the irreducible character $\delta(q)$ of $\mathcal{I}_q$ afforded by $e_{\delta(q),q}$ is independent of q.*
  - (e) *For all $q \in U$ the set $\{e_{\delta(q),q}\}_{\delta(q_0)\in\Delta_{W_0 r_0}(\mathcal{R},q_0)}$ is the complete set of mutually inequivalent primitive central idempotents of $\mathcal{I}_q$.*

(ii) *The continuous families of primitive central idempotents $U \ni q \to e_{\delta(q),q}$ (with $\delta(q_0) \in \Delta_{W_0 r_0}(\mathcal{R},q_0)$) define, for all $q \in U$, a canonical bijection $\delta(q_0) \to \delta(q)$ between the set $\Delta_{W_0 r_0}(\mathcal{R},q_0)$ and the union*

$$\bigcup_{W_0 r \in \mathcal{P}(r_0)} \Delta_{W_0 r(q)}(\mathcal{R},q) \tag{47}$$

Proof. Using the notations of the previous Theorem, we define for all $q \in U_{\epsilon'}$ and for all i:

$$e^{(i)}(q) := \sum_j m^{(i)}_{j,j}(q) \tag{48}$$

This is a primitive central idempotent in $\mathcal{I}_q$ which is independent of the choices of the matrix units $m^{(i)}_{j,k}(q)$. Indeed, another choice of the matrix units would lead to a central primitive idempotent norm close to $e^{(i)}(q)$. This implies unitary equivalence in the C^*-algebra $\mathcal{I}_q$ of these idempotents, but since these idempotents are also central unitary equivalence means actual equality. It follows from this argument that the family of central primitive idempotents $U_{\epsilon'} \ni q \to e^{(i)}(q)$ is continuous at q_0 in the following sense: The family of bounded operators $U_{\epsilon'} \ni q \to \lambda(e^{(i)}(q), q)$ on $L^2(\mathcal{H}(\mathcal{R}, q) = L^2(W)$ is continuous at q_0. Using the independence of the central primitive idempotents for the choice of the matrix units we may repeat this arguments for any $q \in U_{\epsilon'}$ to prove that the families $U_{\epsilon'} \ni q \to e^{(i)}(q)$ are continuous on $U_{\epsilon'}$. If we put $U := U_{\epsilon'}$ it is now straightforward to prove the listed properties of (a)-(e) for the constructed continuous families $e^{(i)}$ of primitive idempotents. Finally the uniqueness follows again from the above rigidity argument for central primitive idempotents, in combination with the continuity, proving (i).

In view of Theorem 3.3(ii) this sets up, for each value of $q \in U$, a bijection between the set of continuous (in the above sense) families of primitive central idempotents $e^{(i)}$ and set of irreducible discrete series characters $\delta(q) \in \Delta_{W_0 r(q)}(\mathcal{R}, q)$ where $W_0 r$ runs over the set $W_0 r \in \mathcal{P}(r_0)$. This proves (ii). □

The above notion of continuity of a q-family of irreducible discrete series characters is special for discrete series characters:

Definition 3.5. *Let $q_0 \in \mathcal{Q}$ and let $\delta_0 \in \Delta(\mathcal{R}, q_0)$. For $q \in U$ (as above) we denote by $\delta(q)$ the equivalence class of irreducible discrete series representations afforded by $e_{\delta(q),q}$. For any open set $U \subset \mathcal{Q}$ we refer to such a family $\delta : q \to \delta(q)$ of equivalence classes of representations afforded by a continuous family of central primitive idempotents in $\mathcal{S}$ (in the above sense, thus in the operator norm of $\mathcal{B}(L^2(W))$) as a "continuous family of irreducible discrete series characters on U". We denote the set of such continuous families by $\Delta(\mathcal{R}, U)$.*

There is also an important weaker notion of continuity for a q-family of characters which is applicable to more general characters:

Definition 3.6. *Let $U \ni q \to \pi(q)$ be a family of equivalence classes of irreducible representations $\pi(q)$ of $\mathcal{Q}(\mathcal{R}, q)$. We say that $q \to \pi(q)$ is a* weakly continuous *family of irreducible characters of $\mathcal{H}(\mathcal{R})$ if $U \ni q \to \chi_{\pi(q)}(N_w)$ is a continuous function for all $w \in W$.*

We denote by $\Delta^{wk}(\mathcal{R}, U)$ be the set of weakly continuous families $U \ni q \to \delta(q)$ of irreducible discrete series characters (i.e. weakly continuous families $q \ni U \to \delta(q)$ such that for all $q \in U$ we have $\chi_{\delta(q)} \in \Delta(\mathcal{R}, q)$).

Continuity of a family of discrete series characters implies weak continuity:

Proposition 3.7. *Let $U \subset \mathcal{Q}$ and let $\delta \in \Delta(\mathcal{R}, U)$. Then the family $q \to \delta(q)$ is also weakly continuous.*

Proof. Indeed, by the Plancherel formula for $\mathcal{H}(\mathcal{R}, q)$ we have

$$\tau(e_{\delta(q),q}) = \deg(\delta(q))\mu_{Pl}(\delta(q)) \tag{49}$$

and hence this function is positive, and continuous by Theorem 3.4(i)(b). Hence the basic formula

$$\chi_{\delta(q)}(N_w) = \deg(\delta(q))\frac{\tau(e_{\delta(q),q}N_w)}{\tau(e_{\delta(q),q})} \tag{50}$$

combined with Theorem 3.4(i)(b),(d) implies the desired continuity. □

Proposition 3.8. *Let $\delta \in \Delta^{wk}(\mathcal{R}, U)$. We define the generic central character map $cc(\delta, \cdot) : U \to W_0\backslash T$ by $cc(\delta, q) = cc(\delta(q))$. Then $cc(\delta)$ is continuous and for all $q \in U$ we have $cc(\delta, q) \in \mathrm{Res}(\mathcal{R}, q)$.*

Proof. This is a trivial consequence of Theorem 2.47 and Proposition 3.7. □

In fact it is true that $cc(\delta) \in W_0\backslash\mathrm{Res}(\mathcal{R})$, but this is not obvious at this point. This result will be shown in Theorem 5.3.

Actually weak continuity and continuity are equivalent for families of discrete series characters. We have:

Theorem 3.9. *Consider the sheaves $\Delta(\mathcal{R})$ and $\Delta^{wk}(\mathcal{R})$ on $\mathcal{Q}$ defined by the presheaves $U \to \Delta(\mathcal{R}, U)$ and $U \to \Delta^{wk}(\mathcal{R}, U)$, respectively.*

(i) *The natural sheaf map $\Delta(\mathcal{R}) \to \Delta^{wk}(\mathcal{R})$ is an isomorphism.*
(ii) *Let $\Delta_{\mathbb{N}}(\mathcal{R})$ denote the sheaf of nonnegative integral linear combinations of $\Delta(\mathcal{R})$, and let $\Delta^{wk}_{\mathbb{N}}(\mathcal{R})$ denote the sheaf of weakly continuous families of (not necessarily irreducible) discrete series characters. The natural map $\Delta_{\mathbb{N}}(\mathcal{R}) \to \Delta^{wk}_{\mathbb{N}}(\mathcal{R})$ is an isomorphism.*

Proof. It is clear that all presheaves involved are sheaves.

Let us prove (i). Given $\delta \in \Delta^{wk}(\mathcal{R}, U)$ we need to show that δ is continuous in the strong sense. Let $q_0 \in U$, and let W_0r_0 be the central character of $\delta(q_0)$. By Theorem 3.4(ii) there exists a neighborhood $V \subset \mathcal{Q}$ of q_0 such that for any $\sigma \in \Delta_{W_0r_0}(\mathcal{R}, q_0)$ there exists $\tilde\sigma \in \Delta(\mathcal{R}, V)$ such that $\sigma = \tilde\sigma_{q_0} := \mathrm{ev}_{q_0}(\tilde\sigma)$ (the evaluation of the strongly continuous family $\tilde\sigma$ at $q_0 \in V$). Moreover Theorem 3.4(ii) asserts that for all $q \in V$ the irreducible discrete series characters $\tilde\sigma_q$ (with $\sigma \in \Delta(\mathcal{R}, q_0)$) are mutually distinct and range over the set of all irreducible discrete series characters of $\mathcal{H}(\mathcal{R}, q)$ whose central character is of the form $W_0r(q)$ for some generic $W_0r \in \mathcal{P}(r_0)$. Now consider $\delta \in \Delta^{wk}(\mathcal{R}, U)$. By Proposition 3.8 it is clear that for all $q \in V$ the central character $cc(\delta(q))$ is of the form $W_0r'(q)$ for some $W_0r' \in \mathcal{P}(r_0)$. The linear independence of irreducible characters, the finiteness of $\Delta_{W_0r_0}(\mathcal{R}, q_0)$ and Proposition 3.7 imply that there exists a finite set $A \subset W$ and a neighborhood $V' \ni q_0$ such that for all fixed $q \in V'$ the finite set of vectors $\Sigma(q) := \{\xi^A_\sigma(q) \in \mathbb{C}^A \mid \sigma \in \Delta_{W_0r_0}(\mathcal{R}, q_0)\}$ with $\xi^A_\sigma(q) := (\chi_{\tilde\sigma_q}(N_w))_{w\in A}$ is linearly independent. In particular the irreducible characters $\tilde\sigma_q$ are separated by the vector $\xi^A_\sigma(q)$ of their values on N_w with $w \in A$. Obviously the maps $\xi^A_\sigma : U \to \mathbb{C}^A$ are continuous. By the weak continuity of δ it follows similarly that the map $\xi^A_\delta : U \to \mathbb{C}^A$ is continuous and by the above, for all $q \in V$ we have $\xi^A_\delta(q) \in \Sigma(q)$. This implies that there exists a unique $\sigma \in \Delta_{W_0r_0}(\mathcal{R}, q_0)$ such that $\delta|_{V'} = \tilde\sigma|_{V'}$, proving that δ is strongly continuous at q_0. Since $q_0 \in U$ was arbitrary the result follows.

Let us now prove (ii). Let $\delta \in \Delta^{wk}_{\mathbb{N}}(\mathcal{R}, U)$. We need to show that δ is continuous in a strong sense. Let $q_0 \in U$, and let W_0r_i (where $i = 1, \ldots, k$) be the set of central characters of the irreducible constituents of $\delta(q_0)$. We have $\delta|_{U^{gen}} = \sum_{W_0r} \delta_{W_0r}|_{U^{gen}}$ (where W_0r runs over the set $W_0\backslash\mathrm{Res}(\mathcal{R})$ of orbits of generic residual points) where

$U^{gen} := \mathcal{Q}^{gen} \cap U$ and where $U^{gen} \ni q \to \delta_{W_0r}(q)$ is a weakly continuous family of discrete series characters such that for all $q \in U^{gen}$, $cc(\delta_{W_0r}(q)) = W_0r(q)$. Recall that $\mathcal{Q}^{gen}$ is the complement of finitely many rational hyperplanes in $\mathcal{Q}$.

We claim that for every connected component $U' \subset U^{gen}$ which contains q_0 in its boundary we have $\delta_{W_0r}|_{U'} \neq 0$ only if $W_0r \in \cup_i \mathcal{P}(r_i)$. Indeed, there exists a $z \in \mathcal{Z}$ such that $z(W_0r_i) = 0$ for $i = 1, \dots, k$ but with $z(W_0r(q_0)) = 1$ for all orbits of generic residual points W_0r such that $W_0r(q_0) \notin \{W_0r_1, \dots, W_0r_k\}$. Observe that for all $r \in \mathrm{Res}(\mathcal{R})$ the value $\deg(\delta_{W_0r}|_{U'}) \in \mathbb{Z}_+$ is independent of $q \in U'$ since the family $\delta_{W_0r}|_{U'}$ is weakly continuous. By the weak continuity of δ on U we see that $U \ni q \to \chi_q := \chi_{\delta(q)}(z)$ must be continuous at q_0; however, by definition of z it follows on the one hand that $\chi_{q_0} = 0$, while on the other hand the limit for $q \to q_0$ from U' yields $\sum_{W_0r \notin \cup_i \mathcal{P}(r_i)} \deg(\delta_{W_0r}|_{U'})$. The claim follows.

We now prove in a similar fashion to the proof in (i) that if $W_0r \in \cup_i \mathcal{P}(r_i)$ and if $U' \subset U^{gen}$ is a connected component which contains q_0 in its boundary then $\delta_{W_0r}|_{U'}$ is strongly continuous and in fact extends uniquely to a neighborhood U'' of q_0 in a strongly continuous sense. This finishes the proof. $\square$

Remark 3.10. *We identify the sheaves $\Delta(\mathcal{R})$, $\Delta^{wk}(\mathcal{R})$, $\Delta_{\mathbb{N}}(\mathcal{R})$ and $\Delta^{wk}_{\mathbb{N}}(\mathcal{R})$ on $\mathcal{Q}$ with their étale spaces. These sheaves are Hausdorff spaces. As sets we have*

$$\Delta(\mathcal{R}) = \coprod_{q \in \mathcal{Q}} \Delta(\mathcal{R}, q) \tag{51}$$

Proof. By Theorem 3.9 it suffices to show this for $\Delta(\mathcal{R})$. In this case the result follows simply from Theorem 3.4(ii). $\square$

Proposition 3.11. *A continuous family of irreducible discrete series characters $U \ni q \to \delta(q)$ is compatible with the scaling maps $\tilde{\sigma}_\epsilon$ (with $\epsilon > 0$) of* [OS, Theorem 1.7] *in the sense that $\tilde{\sigma}_\epsilon(\delta(q)) = \delta(q^\epsilon)$.*

Proof. We may assume that $U \subset \mathcal{Q}$ is an open ball centered around of $q_0 \in \mathcal{Q}$ such that $\mathrm{ev}_{q_0} : \Delta(\mathcal{R}, U) \to \Delta(\mathcal{R}, q_0)$ is an isomorphism. Let $\mathcal{L} \subset \mathcal{Q}$ be the half line generated by q_0. Let $\delta \in \Delta(\mathcal{R}, q_0)$ and let $\tilde{\delta} \in \Delta(\mathcal{R}, U)$ be such that $\mathrm{ev}_{q_0}(\tilde{\delta}) = \delta$. Consider the continuous family $\delta^{(1)}$ defined by restricting the section $\tilde{\delta}$ to $\mathcal{L} \cap U$, and the continuous family $\delta^{(2)}$ defined by scaling $\mathcal{L} \cap U \ni q_0^\epsilon \to \tilde{\sigma}_\epsilon(\delta)$. It follows from the analyticity ([OS, Theorem 1.7], property 1)) that $\delta^{(2)} \in \Delta^{wk}(\mathcal{R}, \mathcal{L} \cap U)$. The result $\delta^{(1)} = \delta^{(2)}$ follows from Theorem 3.9. $\square$

Corollary 3.12. *We can extend any continuous family of irreducible discrete series characters $\delta \in \Delta(\mathcal{R}, U)$ in a unique way to $\tilde{\delta} \in \Delta(\mathcal{R}, \tilde{U})$ where $\tilde{U} = \cup_{\epsilon > 0} U^\epsilon$ is the open cone in $\mathcal{Q}$ generated by U.*

Proof. Let $\mathcal{L} \subset \tilde{U}$ be a half line. By the above Proposition and the properties of the scaling maps (namely, for $\epsilon > 0$ these maps induce bijections of the sets of equivalence classes of irreducible discrete series characters) we see that the restriction $\Delta_{\mathcal{L}}(\mathcal{R})$ of $\Delta(\mathcal{R})$ to $\mathcal{L}$ is a constant sheaf. The result follows easily from this remark. $\square$

4. The generic formal degree

Let $U \subset \mathcal{Q}$ be a connected open cone, and let $\delta \in \Delta^{wk}(\mathcal{R}, U)$. In this subsection we prove the rationality of the formal degree $U \ni q \to \mu_{Pl}(\delta(q))$, i.e. we prove that this function is the restriction to U of a rational function of the root parameters

$q_{\alpha^\vee}$ with rational coefficients, i.e. of an element of $K(\Lambda_{\mathbb{Z}})$. We refer to this rational function as the *generic formal degree* of the family δ. We combine the rationality of the generic formal degree with the product formula [O3, Theorem 4.10] for the formal degree of $\delta(q)$ valid for q varying in a half line in $\mathcal{Q}$. We then obtain the factorization of the generic formal degree as element of $K(\Lambda)$.

4.1. **Rationality of the generic formal degree.** Let $\mathcal{R}$ be a semisimple root datum and let $\Omega \subset W$ be the subgroup of length zero elements. If f is a facet of the fundamental alcove C we denote by $\Omega_f \subset \Omega$ be the stabilizer of f in Ω. Let $\langle f \rangle \subset E$ be the affine subspace spanned by f, and let $E/\langle f \rangle$ be the linear space formed by cosets $e - \langle f \rangle$ (with $e \in E$) of the linear subspace associated to $\langle f \rangle$. Let ϵ_f be the determinant character of the linear action of Ω_f on $E/\langle f \rangle$. The involutive subalgebras $\mathcal{H}(\mathcal{R}, f, q) = \mathcal{H}(W_f, q) \rtimes \Omega_f \subset \mathcal{H}(\mathcal{R}, q)$ are finite dimensional (since we assume that $\mathcal{R}$ is semisimple here) and hence semisimple (compare with [OS, Lemma 1.4]). We start with a useful lemma:

Lemma 4.1. *The algebra $\mathcal{H}(\mathcal{R}, f, q)$ is semisimple for all $q \in \mathcal{Q}$. Let F be an algebraic closure of $K(\Lambda)$ and let $\mathcal{H}_\Lambda(\mathcal{R}, f)$ be the generic algebra over Λ. We denote by $I \supset \Lambda$ the integral closure of Λ in F. Extend $ev_1 : \Lambda \to \mathbb{C}$ to I and let $\chi \leftrightarrow \chi^F$ be the corresponding bijection between $\widehat{W_f \rtimes \Omega_f}$ and $\widehat{\mathcal{H}_F(\mathcal{R}, f)}$ (as in* [Car, Proposition 10.11.4]*). Let $d_\chi \in F$ denote the formal degree of χ^F with respect to the trace form τ restricted to the algebra $\mathcal{H}_F(\mathcal{R}, f)$. Then $d_\chi \in K(\Lambda)$ and d_χ is regular on $\mathcal{Q}$ for all $\chi \in \widehat{W_f \rtimes \Omega_f}$.*

Proof. For all $q \in \mathcal{Q}$ the trace form τ of the algebra $\mathcal{H}(\mathcal{R}, f, q)$ has a nonzero discriminant, proving that $\mathcal{H}(\mathcal{R}, f, q)$ (and a fortiori $\mathcal{H}_F(\mathcal{R}, f)$) is a symmetric (and thus semisimple) algebra. Let (V, σ^F) be a matrix representation of $\mathcal{H}_F(\mathcal{R}, f)$ whose character equals χ^F. We write $d_\sigma := d_\chi$ for its formal degree (with respect to τ).

The orthogonality of characters of a symmetric algebra implies that

$$d_\sigma = 1/S_\sigma \tag{52}$$

where S_σ is the Schur element of σ^F, given by

$$\dim_F(V) S_\sigma = \sum_{w \times \omega \in W_f \rtimes \Omega_f} \chi^F(N_{w \times \omega}) \chi^F(N_{(w \times \omega)^{-1}}) \tag{53}$$

By a well known result (see e.g. the argument in [G, Proposition 4.6], which applies to our situation as well as one easily checks) one also has the following formula for the Schur element:

$$\dim_F(V) S_\sigma(q) \, \mathrm{Id_V} = \sum_{w \times \omega \in W_f \rtimes \Omega_f} \sigma^F(N_{w \times \omega} N_{(w \times \omega)^{-1}}) \tag{54}$$

But clearly (loc. cit.)

$$\sum_{w \times \omega \in W_f \rtimes \Omega_f} \sigma^F(N_{w \times \omega} N_{(w \times \omega)^{-1}}) = |\Omega_f| \sum_{w \in W_f} \sigma^F(N_w N_{w^{-1}}) \tag{55}$$

This last equality implies that if (σ_1^F, V_1) is any simple submodule of the restriction of σ^F to $\mathcal{H}_F(W_f)$ then

$$\dim_F(V) S_\sigma = |\Omega_f| \dim_F(V_1) S_{\sigma_1} \tag{56}$$

The right hand side of this equation is known to be in $K(\Lambda)$, proving the desired result. The last assertion follows from the well known fact that the Schur element of S_σ is nonzero at q iff σ^F corresponds to a projective irreducible representation of the specialization algebra $\mathcal{H}(W_f, q)$. Since $\mathcal{H}(W_f, q)$ is semisimple for $q \in \mathcal{Q}$ this holds true for all σ. $\square$

Let $\delta \in \Delta^{wk}(\mathcal{R}, U)$. Following [ScSt], [R1] we define for $q \in U$ the index function $f_{\delta,q} \in \mathcal{H}(\mathcal{R}, q)$ by

$$f_{\delta,q} = \sum_f (-1)^{\dim(f)} \sum_{\sigma\in\mathrm{Irr}(\mathcal{H}(\mathcal{R},f,q))} \deg(\sigma)^{-1}[\delta_q|_{\mathcal{H}(\mathcal{R},f,q)} \otimes \epsilon_f : \sigma]e_\sigma \in \mathcal{H}(\mathcal{R}, q) \tag{57}$$

where f runs over a complete set of representatives of the Ω-orbits of faces of the fundamental alcove C, and where $e_\sigma \in \mathcal{H}(\mathcal{R}, f, q)$ denotes the primitive central idempotent in the finite dimensional complex semisimple algebra $\mathcal{H}(\mathcal{R}, f, q)$ affording σ. The importance of the element $f_{\delta,q} \in \mathcal{H}(\mathcal{R}, q)$ is that it links character theory with the elliptic pairing. Indeed, following [ScSt], [R1] one shows, using the Euler-Poincaré principle and Frobenius reciprocity, that for all representations π of finite length of $\mathcal{H}(\mathcal{R}, q)$ one has (see [OS, Proposition 3.6]):

$$\chi_\pi(f_{\delta,q}) = EP_{\mathcal{H}}(\delta(q), \pi) \tag{58}$$

Definition 4.2. *The multiplicities* $[\delta(q)|_{\mathcal{H}(\mathcal{R},f,q)} \otimes \epsilon_f : \sigma]$ *are independent of* $q \in U_\delta$ *by Proposition 3.7. We denote these multiplicities by* $[\delta_f \otimes \epsilon_f : \sigma] \in \mathbb{Z}_{\geq 0}$

Theorem 4.3. *Let* $U \subset \mathcal{Q}$ *be a connected open cone and let* $\delta \in \Delta^{wk}(\mathcal{R}, U)$. *We have the following index formula for the formal degree* $\mu_{Pl}(\{\delta(q))\})$ *(with* $q \in U$*):*

$$\mu_{Pl}(\{\delta(q)\}) = \tau(f_{\delta,q}) = \sum_f (-1)^{\dim(f)} \sum_{\sigma\in\mathrm{Irr}(\mathcal{H}(\mathcal{R},f,q))} [\delta_f \otimes \epsilon_f : \sigma]d_\sigma(q) \tag{59}$$

Here f *runs over a complete set of representatives of the* Ω*-orbits of faces of* C, *and* $d_\sigma(q)$ *denotes the formal degree of* σ *in the finite dimensional Hilbert algebra* $\mathcal{H}(\mathcal{R}, f, q)$ *(as in Lemma 4.1).*

Proof. We apply the Plancherel formula (28) to $f_{\delta,q}$. In view of (58) and Corollary 2.34 we see that $\mu_{Pl}(\{\delta(q)\}) = \tau(f_{\delta,q})$. Now use (57) and Definition 4.2. $\square$

Corollary 4.4. *Let* $U \subset \mathcal{Q}$ *be a connected open cone and let* $\delta \in \Delta^{wk}(\mathcal{R}, U)$. *The formal degree* $U \ni q \to \mu_{Pl}(\{\delta(q)\})$ *is the restriction to* U *of a rational function in the parameters* $q_{\alpha^\vee}$ *(with* $\alpha \in R_{\mathrm{nr}}$*) with rational coefficients (or in other words, an element of* $K(\Lambda_{\mathbb{Z}})$ *in the notation of Proposition 2.62(ii)). This rational function is regular on* $\mathcal{Q}$ *and positive on* U.

Proof. Consider the index formula as given in Theorem 4.3. The result now follows from Lemma 4.1 (the positivity on U is obvious). $\square$

4.2. Factorization of the generic formal degree.

Lemma 4.5. *Let* $\delta \in \Delta^{wk}(\mathcal{R}, U)$ *be a weakly continuous family of irreducible discrete series characters on a convex open cone* $U \subset \mathcal{Q}$. *The map* $cc(\delta(\cdot)) : U \to W_0\backslash T$ *is continuous. There exist finitely many mutually disjoint, nonempty connected open subcones* $U_i \subset U$ *such that* $\cup_i U_i \subset U$ *is dense, and such that for each* i *there exists an orbit* $W_0 r_i$ *of generic residual cosets such that* $\overline{U_i} \cap U \subset \mathcal{Q}^{gen}_{W_0 r_i}$ *and* $cc(\delta)|_{U_i} = W_0 r_i|_{U_i}$. *In particular* $cc(\delta)$ *is continuous and piecewise analytic.*

Proof. The continuity of $cc(\delta)$ on U follows from Proposition 3.8. Let U_i run over the finite set of connected components of $U \cap \mathcal{Q}^{gen}$. Then the restriction of $cc(\delta)$ to U_i must coincide with the restriction of a unique orbit of generic residual points, by the continuity of $cc(\delta)$ and the definition of $\mathcal{Q}^{gen}$. By continuity, for all $q \in \overline{U}_i \cap U$ the orbit $W_0 r_i(q)$ carries discrete series representations. Hence $r_i(q)$ is residual, or equivalently $q \in \mathcal{Q}^{reg}_{W_0 r_i}$. □

Theorem 4.6. *Let $\delta \in \Delta^{wk}(\mathcal{R}, U)$ be a weakly continuous family of irreducible discrete series characters on a convex open cone $U \subset \mathcal{Q}$. Let r be a generic residual point such that there exists a nonempty connected open subcone $U_i \subset U$ such that $cc(\delta)|U_i = W_0 r|U_i$ (see Lemma 4.5). There exists a constant $d \in \mathbb{Q}^\times$ (depending on δ and $W_0 r$) such that we have the following equality in $K(\Lambda_\mathbb{Z})$:*

$$\mu_{Pl}(\{\delta\}) = d m_{W_0 r} \tag{60}$$

Here $m_{W_0 r} \in K(\Lambda_\mathbb{Z})$ (see Proposition 2.62(ii)) is the function defined in (40).

Proof. We fix $f_s \in \mathbb{R}$ and we denote the corresponding half line in $\mathcal{Q}$ by $\mathcal{L} \subset \mathcal{Q}$ (see Remark 2.49). Notice that either $\mathcal{L} \cap U_i = \emptyset$ or $\mathcal{L} \subset U_i$; assume that $\mathcal{L}$ is such that we are in the latter situation. By [O1, Corollary 3.32, Theorem 5.6] we have

$$\mu_{Pl}(\{\delta(q)\}) = d(q) m_{W_0 r}(q) \tag{61}$$

for all $q \in U_i$, where $d(q) \in \mathbb{R}^\times$ has the property that for all $\epsilon \in \mathbb{R}_+$

$$d(q^\epsilon) = d(q) \tag{62}$$

where q^ϵ is defined by $q^\epsilon(s) = (q(s))^\epsilon$ for all affine simple reflections s. By Theorem 2.60, Corollary 4.4 and (61) we see that d is itself a rational function which is regular on U_i.

Recall that we view $\mathbf{q} > 1$ as coordinate on $\mathcal{L}$. The expressions $\alpha(r(q)) = \alpha(s)\alpha(c(q))$ and $q_{\alpha^\vee}$ (with $\alpha \in R_{\mathrm{nr}}$ and $q \in \mathcal{L}$) are thus viewed as functions of $\mathbf{q} > 1$. By the form of the right hand side of (61) as given in (59), and in view of Corollary 4.4 we see that there exists a unique real number f such that

$$\lim_{\mathbf{q} \to \infty} \mathbf{q}^f \mu_{Pl}(\{\delta\})(\mathbf{q}) = a_\mathcal{L} \in \mathbb{Q}^\times \tag{63}$$

On the other hand, by (62) the rational function d has a constant value, $d_\mathcal{L}$ say, on $\mathcal{L}$. Hence (61) implies, in view of (40) and Proposition 2.62(ii), that $d_\mathcal{L} b_\mathcal{L} = a_\mathcal{L}$ where

$$\lim_{\mathbf{q} \to \infty} \mathbf{q}^f m_{W_0 r}(\mathbf{q}) = b_\mathcal{L} \in \mathbb{Q}^\times \tag{64}$$

Since $d(q)$ is continuous as a function of $q \in U_i$ this implies that $d_\mathcal{L} \in \mathbb{Q}$ is independent of $\mathcal{L} \subset U_i$ and thus that $d(q) = d$ is independent of $q \subset U_i$. Since U_i is an open set, the equality (60) of rational functions which we have now proved on U_i extends to $\mathcal{Q}$ (recall that both sides are regular on $\mathcal{Q}$). □

Corollary 4.7. *Let $\delta \in \Delta^{wk}(\mathcal{R}, U)$ be weakly continuous on a convex open cone U. Let $W_0 r_i$, $W_0 r_j$ be orbits of generic residual points associated with δ as in Lemma 4.5. There exists a constant $d \in \mathbb{Q}^\times$ such that $m_{W_0 r_i} = d m_{W_0 r_j}$.*

5. The generic central character map and the formal degrees

The following result depends on the classification of residual points:

Lemma 5.1. *Let $\mathcal{R} = (X, R_0, Y, R_0^\vee)$ be a simple root datum such that R_0 is not simply laced, and let $r, r' \in \mathrm{Res}(\mathcal{R})$ be generic residual points with equal unitary part s which is W_0-invariant. If there exists a constant $d \in \mathbb{C}^\times$ such that $m_{W_0r} = dm_{W_0r'}$ then $W_0r = W_0r'$.*

Proof. Using Lemma 2.53 and Proposition 2.62(iii) we reduce to the case where $\mathcal{R}$ is irreducible and $X = P(R_1)$, and r, r' are generic residual points with equal W_0-invariant unitary part $s \in T_u$. Let us write $r = sc$ and $r' = sc'$. In the $C_n^{(1)}$-case we have $s = (1, \dots, 1)$ or $s = (-1, \dots, -1)$. We use Proposition 2.56. In the first case we find that c, c' extend to positive generic residual points for the root datum $\mathcal{R}'$ defined by $R_0' = B_n$ and $X' = P(R_0)$, with the parameters $\tilde{q}$ defined by $\tilde{q}_{e_i \pm e_i} = q_{e_i \pm e_j}$ and $\tilde{q}_{2e_i} = q_{2e_i}^{1/2} q_{2e_i+1}^{1/2}$. In the second case c, c' are positive generic residual points for $\mathcal{R}'$ with the parameter $\tilde{q}$ defined by $\tilde{q}_{e_i \pm e_i} = q_{e_i \pm e_j}$ and $\tilde{q}_{2e_i} = q_{2e_i}^{-1/2} q_{2e_i+1}^{1/2}$. In the first case we substitute $q_{2e_i} = q_{2e_i+1}$ and in the second case we substitute $q_{2e_i} = q_{2e_i+1}^{-1}$; with this substitution we have in either case

$$m_{W_0r}^{\mathcal{R}}(q) = m_{W_0c}^{\mathcal{R}'}(\tilde{q}) \text{ and } m_{W_0r'}^{\mathcal{R}}(q) = m_{W_0c'}^{\mathcal{R}'}(\tilde{q}) \tag{65}$$

Therefore it suffices to prove the assertion for irreducible root data $\mathcal{R}$ such that R_0 is non-simply laced and $X = P(R_0)$ where W_0r, W_0r' are orbits of generic residual points with the same W_0-invariant unitary part s. We may now replace s by 1 without loss of generality. Hence we may and will assume that W_0r, W_0r' are orbits of positive residual points. We again use Proposition 2.56 to compare such points to the classification in [HO1, Section 4].

In the cases G_2 and F_4 the W_0-orbit W_0r of a generic positive residual points W_0r is distinguished by the set $\mathcal{Q}_{W_0r}^{\mathrm{reg}}$ as can be seen from Tables 2 and 4. Since this set is the complement of the zero set of m_{W_0r} (by Theorem 2.60) the desired conclusion follows.

Next consider the cases B_n and C_n. Let f be a rational function in q_1, q_2 of the form

$$f(q) = q_1^{N_1} q_2^{N_2} \prod_i \prod_{j \geq 0} (q_1^i q_2^j - 1)^{n_{i,j}} \tag{66}$$

(with $n_{i,j} \in \mathbb{Z}$). Then the exponents $n_{i,j} \in \mathbb{Z}$ are determined by f. Let q_1 denote the parameter of the roots $\pm e_i \pm e_j$ and q_2 the parameter of $\alpha^\vee$ for $\alpha = e_i$ (if R_0 has type B_n) or $2e_i$ (if R_0 has type C_n). The functions m_{W_0r} are of all of the above form where the exponent of q_2 is 0, 2 or 4. The W_0-orbits of generic positive residual points are parametrized by partitions of n (see [HO1, Section 4], and [O3, Theorem A.7]). Let $\lambda \vdash n$ and let W_0r_λ be the corresponding W_0-orbit of residual points. Let us use the notation $m_{W_0r} = m_\lambda$ if $W_0r = W_0r_\lambda$. In the case B_n, the factors of m_λ of the form $(q_1^{2i} q_2^2 - 1)$ have multiplicity $n_{2i,2}$ equal to twice the number of boxes $b \in \lambda$ such that $c(b) = i$ (where $c(b)$ denotes the content of b). Hence m_λ determines for each i the number of boxes in λ with content i. Clearly this determines λ. If R_0 is of type C_n we use the correspondence between B_n and C_n positive generic residual points as explained in the proof of Theorem 2.58. It follows that the factors of m_λ

of type $(q_1^{4i}q_2^2 - 1)$ have multiplicity $n_{4i,2}$ equal to twice the number of boxes b of λ with $c(b) = i$, and again we conclude that λ is determined by m_λ. $\square$

Corollary 5.2. *Let $\mathcal{R}$ be semisimple and let $q_0 \in \mathcal{Q} = \mathcal{Q}(\mathcal{R})$. Suppose that $\delta_0 \in \Delta(\mathcal{R}, q_0)$ and that $cc(\delta_0) = W_0 r_0$ for a $r_0 \in \mathrm{Res}^s(\mathcal{R}, q_0)$ with $s \in T_u$ which is W_0-invariant. Then there exists a unique orbit $W_0 r \in W_0 \backslash \mathrm{Res}(\mathcal{R})$ of generic residual points which has the following property: there exists an open neighborhood $U \subset \mathcal{Q}$ of q_0 and a continuous family of discrete series characters $U \ni q \to \delta(q) \in \Delta_{W_0 r(q)}(\mathcal{R}, q)$ such that $cc(\delta(q)) = W_0 r(q)$ for all $q \in U$.*

Proof. The uniqueness of such an orbit $W_0 r$ of generic residual points is clear from the fact that a generic residual point is real analytic on $\mathcal{Q}$. Hence $W_0 r$ is determined by its restriction to U.

For existence we first choose a lift $\tilde{r}_0 \in \mathrm{Res}(\mathcal{R}^{max}, q_0)$ of r_0 and a $\pi_0 \in \Delta_{W_0 \tilde{r}_0}(\mathcal{R}, q_0)$ with the property that δ_0 is a component of the restriction of π_0 to $\mathcal{Q}(\mathcal{R}, q_0)$. According to Theorem 3.4 there exists an open neighborhood $U \subset \mathcal{Q}$ such that π_0 extends to a continuous family π of irreducible discrete series characters of $\mathcal{H}(\mathcal{R}^{max})$. It is obvious that $\pi = \pi^{(1)} \otimes \cdots \otimes \pi^{(m)}$ with $\pi^{(i)}$ a continuous family of irreducible discrete series characters of $\mathcal{H}(\mathcal{R}^{(i)})$ defined on $U^{(i)}$ (where $\mathcal{R}^{(i)}$ with $i = 1, \ldots, m$ runs through the simple factor of $\mathcal{R}^{max}$ as in Proposition 2.37).

For each i there exists a generic residual point $\tilde{r}^{(i)} \in \mathrm{Res}(\mathcal{R}^{(i)})$ such that $cc(\pi^{(i)}) = W(R_0^{(i)})\tilde{r}^{(i)}$ on $U^{(i)}$. Indeed, if $\mathcal{R}^{(i)}$ is simply laced then this is trivial by the scaling isomorphisms [OS, Theorem 1.7(1),(5)]. So let us assume that $\mathcal{R}^{(i)}$ is not simply laced. Then the assertion follows from Lemma 4.5, Theorem 4.6, and Lemma 5.1 applied to

$$\pi_0^{(i)} \in \Delta_{W(R_0^{(i)})\tilde{r}_0^{(i)}}(\mathcal{R}^{(i)}, q_0^{(i)}) \tag{67}$$

Let $r \in \mathrm{Res}(\mathcal{R})$ be the generic residual point that corresponds to $(\tilde{r}^{(1)}, \ldots, \tilde{r}^{(m)})$ by restriction as in Lemma 2.53(i).

If we restrict the continuous family π from $\mathcal{H}(\mathcal{R}^{max})$ to $\mathcal{H}(\mathcal{R})$ we obtain a continuous family of discrete series characters, i.e. a section $\pi|_{\mathcal{H}(\mathcal{R})} \in \Delta_{\mathbb{N}}(\mathcal{R}, U)$. Observe that all irreducible components of $\pi(q)|_{\mathcal{H}(\mathcal{R},q)}$ have the same central character. Using the linear independence of irreducible characters and Theorem 3.4(ii) we see that $\pi|_{\mathcal{H}(\mathcal{R})}$ contains the continuous extension δ of δ_0 to U with multiplicity at least 1. In particular we see that the composition of $cc(\pi) : U \to W_0 \backslash T^{max}$ with the natural projection $W_0 \backslash T^{max} \to W_0 \backslash T$ is the central character $cc(\delta)$ of the family δ on U. We conclude that $cc(\delta)$ is given on U by $W_0 r|_U$, where $r \in \mathrm{Res}(\mathcal{R})$ was constructed above. This finishes the proof. $\square$

The next result the main result of this section. It generalizes Corollary 5.2 to general irreducible discrete series characters.

Theorem 5.3. *Let $\delta_0 \in \Delta(\mathcal{R}, q_0)$. Let $U \subset \mathcal{Q}$ be a (connected) open neighborhood of q_0 such that there exists a $\delta \in \Delta(\mathcal{R}, U)$ with $\delta(q_0) = \delta_0$ (see Theorem 3.4). There exists a unique orbit $W_0 r \in W_0 \backslash \mathrm{Res}_q(\mathcal{R})$ such that $cc(\delta(\cdot)) = W_0 r|_U$.*

Proof. We first show that the notion of weak continuity of a family of characters (see Definition 3.6) is to some extent compatible with the reduction results Theorem 2.6 and Corollary 2.28.

Let $cc(\delta(q)) = W_0 t(q)$ where $U \ni q \to t(q) \in T$ is continuous. Write s for the unitary part of $t(q)$ (independent of q). Let $\psi_s : \mathcal{Q} \to \mathcal{Q}_s = \mathcal{R}_s$ be the homomorphism given by $q \to q_s$.

We denote by $\pi_0 \in \Delta_{\mathbb{N}}(\mathcal{R}_s, \psi_s(q_0))$ the restriction of the irreducible discrete series module of $\mathcal{H}(\mathcal{R}_s, \psi_s(q_0)) \rtimes \Gamma(t(q_0))$ to $\mathcal{H}(\mathcal{R}_s, \psi_s(q_0))$. By Theorem 3.4 and Theorem 3.9 there exists a (connected) open neighborhood $U_s \subset \mathcal{Q}_s$ of $\psi_s(q_0)$ and a family

$$\pi \in \Delta_{\mathbb{N}}(\mathcal{R}_s, U_s) \tag{68}$$

such that $\pi(\psi(q_0)) = \pi_0$. We may and will shrink U in such a way that $\psi_s(U) \subset U_s$.

Let $N^s_w \in \mathcal{H}(\mathcal{R}_s, q_s)$ for $w \in W(\mathcal{R}_s)$ denote the standard basis for the affine Hecke algebra $\mathcal{H}(\mathcal{R}_s, q_s)$. Recall from Lusztig's construction (in the variation Theorem 2.6) that $\mathcal{H}(\mathcal{R}_s, q_s)$ is embedded as a subalgebra of the formal completion $\overline{\mathcal{H}}(\mathcal{R}, q)$ (as defined by (21)) via the map $N^s_w \to e_{t(q)} N_w$ where $w \in W(\mathcal{R}_s)$ and where $e_{t(q)} \in \overline{\mathcal{H}}(\mathcal{R}, q)$ denotes the idempotent as in Theorem 2.6.

Let $\delta_t(q)$ be the irreducible discrete series representation of $\overline{\mathcal{H}}(\mathcal{R}_s, q_s) \rtimes \Gamma(t(q))$ corresponding to $\delta(q)$ according to Theorem 2.6. This implies in particular that

$$\chi_{\delta_t(q)}(N^s_w) = \chi_{\delta(q)}(e_{t(q)} N_w) \tag{69}$$

for all $w \in W(\mathcal{R}_s)$.

We claim that

$$\chi_{\pi(q_s)}(N^s_w) = \chi_{\delta_t(q)}(N^s_w) \tag{70}$$

for all $q \in U$ and $w \in W(\mathcal{R}_s)$. By Theorem 3.4 and Theorem 3.9 it suffices to show that for all $w \in W$ the right hand side of (69) is continuous as a function of $q \in U$.

By the continuity of $U \ni q \to cc(\delta(q))$ it is easy to see that one can construct for each $N \in \mathbb{N}$ a continuous family $U \ni q \to a_{t,q} \in \mathcal{A} = \mathbb{C}[T]$ (i.e. a q-family of Laurent polynomials on T whose coefficients depend continuously on q) such that for all $q \in U$ and $t' \in W(R_{s,1})t(q)$: $a_{t,q} \in 1 + m^N_{t'}$ while for all $t' \in W_0 t(q) \backslash W(R_{s,1})t(q)$ one has $a_{t,q} \in m^N_{t'}$. If N is sufficiently large this implies easily that for all $q \in U$ and for any $w \in W(\mathcal{R}_s)$ one has

$$\chi_{\delta(q)}(e_{t(q)} N_w) = \chi_{\delta(q)}(a_{t,q} N_w) \tag{71}$$

which is indeed continuous in $q \in U$ as was required, thus proving (70).

According to Corollary 5.2 we find that $cc(\pi_\lambda) \in W(R_{s,1}) \backslash \mathrm{Res}^s(\mathcal{R}_s)$ for any irreducible component π_λ of π. By relation (70) and application of Corollary 2.54 it follows that for any component π_λ of π that

$$cc(\delta) = (\Phi^{W_0}_{W_0 s})^{-1}(\Gamma_s(cc(\pi_\lambda))) \tag{72}$$

This finishes the proof. $\square$

In view of Theorem 2.58 this means that the central character of $\delta \in \Delta(\mathcal{R}, U)$ actually extends to a $\mathcal{Q}_c$-valued point of $W_0 \backslash T$.

Definition 5.4. *(Generic central character for discrete series) Let $q \in \mathcal{Q}$. Theorem 5.3 yields a map $gcc_q : \Delta(\mathcal{R}, q) \to W_0 \backslash \mathrm{Res}_q(\mathcal{R})$ which extends to a continuous map (in the sense of Remark 3.10) $gcc : \Delta(\mathcal{R}) \to W_0 \backslash \mathrm{Res}(\mathcal{R})$. We call gcc_q and gcc the "generic central character" maps.*

Definition 5.5. *Consider the topological space $\mathcal{O}(\mathcal{R})$ given by*

$$\mathcal{O}(\mathcal{R}) = \{(W_0r, q) \in W_0\backslash \operatorname{Res}(\mathcal{R}) \times \mathcal{Q} \mid q \in \mathcal{Q}^{reg}_{W_0r}\} \tag{73}$$

Then $\pi_2 : \mathcal{O}(\mathcal{R}) \to \mathcal{Q}$ is a local homeomorphism and the projection

$$\pi_1 : \mathcal{O}(\mathcal{R}) \to W_0\backslash \operatorname{Res}(\mathcal{R}) \tag{74}$$

on the first factor defines for all $q \in \mathcal{Q}$ a bijection between the fibre $\mathcal{O}(\mathcal{R})_q$ of π_2 at $q \in \mathcal{Q}$ and the set $W_0\backslash \operatorname{Res}_q(\mathcal{R})$. We define the following evaluation map:

$$\begin{aligned} \mathrm{ev} : \mathcal{O}(\mathcal{R}) &\to W_0\backslash T \times \mathcal{Q} \\ (W_0r, q) &\to (W_0r(q), q) \end{aligned}$$

The generic central character map of Definition 5.4 can be characterized as follows:

Proposition 5.6. *We define $GCC = gcc \times \pi : \Delta(\mathcal{R}) \to \mathcal{O}(\mathcal{R})$ where $\pi : \Delta(\mathcal{R}) \to \mathcal{Q}$ is the canonical projection. Then GCC is the unique continuous map such that the following diagram commutes:*

$$\begin{array}{ccc} \Delta(\mathcal{R}) & \xrightarrow{GCC} & \mathcal{O}(\mathcal{R}) \\ {\scriptstyle cc_\Lambda}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{ev}} \\ W_0\backslash T \times \mathcal{Q} & = & W_0\backslash T \times \mathcal{Q} \end{array} \tag{75}$$

Proof. This is a reformulation of Theorem 5.3. ☐

We are now in the position to formulate the first main result of this paper:

Theorem 5.7. *The map $GCC = gcc \times \pi : \Delta(\mathcal{R}) \to \mathcal{O}(\mathcal{R})$ is a surjective local homeomorphism and gives $\Delta(\mathcal{R})$ the structure of a locally constant sheaf on $\mathcal{O}(\mathcal{R})$.*

Proof. As a consequence of the definition of gcc in Definition 5.4 we can reformulate Theorem 3.4(ii) by stating that for any $W_0r \in W_0\backslash \operatorname{Res}(\mathcal{R})$ and any connected component $U \subset \mathcal{Q}^{\mathrm{reg}}_{W_0r}$ the restriction $\Delta_C(\mathcal{R})$ of Δ to the connected component $C = \{W_0r\}\times U \subset \mathcal{O}(\mathcal{R})$ is a locally constant sheaf on C. In particular the cardinality of the fibres of $GCC|_{\Delta_C(\mathcal{R})}$ is constant. Hence the surjectivity of GCC follows from Theorem 2.47 by considering a generic parameter $q \in U$. ☐

Corollary 5.8. *Let $W_0r \in W_0 \operatorname{Res}(\mathcal{R})$ and let $U \subset \mathcal{Q}^{\mathrm{reg}}_{W_0r}$ be a connected component as in the proof of Theorem 5.7. The restriction $\Delta_C(\mathcal{R})$ of Δ to the component $C = \{W_0r\} \times U \subset \mathcal{O}(\mathcal{R})$ of $\mathcal{O}(\mathcal{R})$ is a constant sheaf.*

Proof. Since U is the interior of a convex polyhedral cone by Theorem 2.60 this follows trivially from Theorem 5.7. ☐

Corollary 5.9. *For all $q \in \mathcal{Q}$ the map $gcc_q : \Delta(\mathcal{R}, q) \to W_0\backslash \operatorname{Res}_q(\mathcal{R})$ is surjective.*

Proof. This follows immediately from the surjectivity of GCC. ☐

In particular, if $\delta_0 \in \Delta(\mathcal{R}, q_0)$ with $gcc_q(\delta_0) = W_0r \in \operatorname{Res}_{q_0}(\mathcal{R})$ is an irreducible discrete series character and $U \subset \mathcal{Q}^{reg}_{W_0r}$ denotes the component of q_0, then there exists a unique continuous family $\delta \in \Delta(\mathcal{R}, U)$ such that $\mathrm{ev}_{q_0}(\delta) = \delta_0$. Observe that the open cone $U \subset \mathcal{Q}$ is the maximal set to which δ can be continued as a discrete series character (since the central character $W_0r(q)$ will cease to be residual at every boundary point of U). Hence the open cone U is determined by δ.

Definition 5.10. *We denote this open cone by U_δ, and we call a continuous family of irreducible discrete series characters δ which is extended to its maximal domain of definition $U_\delta \ni q \to \delta(q)$ a* generic irreducible discrete series character*. We denote by $\Delta^{gen}(\mathcal{R})$ the finite set of generic irreducible discrete series characters.*

Corollary 5.11. *For each component $C = \{W_0 r\} \times U$ of $\mathcal{O}(\mathcal{R})$ we define a multiplicity $M_C \in \mathbb{Z}_{\geq 0}$ of C by $M_C := |\{\delta \in \Delta^{gen}(\mathcal{R}) \mid GCC(\delta) = C\}|$. Then $M_C > 0$ for all components $C = \{W_0 r\} \times U$). For all $q \in U$ one has $M_C = |\Delta_{W_0 r}(\mathcal{R}, q)|$, and for all $q \in \mathcal{Q}$ one has (where χ_U denotes the characteristic function of U):*

$$|\Delta(\mathcal{R}, q)| = \sum_{W_0 r \in W_0 \backslash \operatorname{Res}(\mathcal{R})} \sum_{U \in \mathcal{C}_{W_0 r}} \chi_U(q) M_{\{W_0 r\} \times U} \tag{76}$$

We reformulate Theorem 4.6 using our results on the generic central character. This is the second main theorem of this paper:

Theorem 5.12. *Let $\delta \in \Delta^{gen}(\mathcal{R})$. There exists a rational constant $d_\delta \in \mathbb{Q}^\times$ such that for all $q \in U_\delta$ we have*

$$\mu_{Pl}(\{\delta(q)\} = d_\delta m_{gcc(\delta)}(q) \tag{77}$$

Here $m_{gcc(\delta)} \in K(\Lambda_\mathbb{Z})$ is explicitly given by (40).

Remark 5.13. *This result proves in particular Conjecture* [O1, 2.27]*, and it shows that the constants defined in Conjecture* [O1, 2.27] *for special values of the parameters can be determined from the rational constants d_δ defined for the irreducible generic discrete series characters. Indeed, any irreducible discrete series character $\delta_0 \in \Delta(\mathcal{R}, q_0)$ determines a unique $\delta \in \Delta^{gen}(\mathcal{R})$ such that $\delta_0 = \delta(q_0)$. The constant defined in Conjecture* [O1, 2.27] *is equal to d_δ multiplied by a rational number depending on q_0 which can be easily expressed in terms of the sets $R^{p,+}_{r,1}$, $R^{p,-}_{r,1}$, and $R^{z}_{r,1}$ of roots whose associated factor in $m_{W_0 r}$ becomes zero at q_0).*

6. The generic linear residual points and the evaluation map

In this section we summarize, following [HO1] and [Slo2], the classification of the W_0-orbits of the generic linear residual points for all irreducible root systems R_1 and we describe the evaluation map at a given parameter $k \in \mathcal{K} = \mathcal{K}(R_1)$ of the parameter space associated with R_1.

For each generic linear residual point ξ of R_1 we will describe the open dense set $\mathcal{K}^{reg}_\xi$ of parameters k such that $\mathrm{ev}_k(\xi) = \xi(k)$ is still residual. In addition we will describe the set $W_0 \backslash \mathrm{Res}^{lin}(R_1, V, k)$ of residual orbits for each $k \in \mathcal{K}$. To do this it is convenient to use the notion of k-weighted distinguished Dynkin diagrams with respect to a given bases $F_1 = \{\alpha_1, \ldots, \alpha_n\}$ of simple roots of R_1:

Definition 6.1. *For $k \in \mathcal{K}$ we define the set $\mathrm{Dyn}^{dist}(R_1, V, F_1, k)$ of distinguished k-weighted Dynkin diagrams for (R_1, V, F_1, k) as the set of F_1-dominant linear (R_1, k)-residual points. There is a canonical bijection*

$$W_0 \backslash \mathrm{Res}^{lin}(R_1, V, k) \xrightarrow{\simeq} \mathrm{Dyn}^{dist}(R_1, V, F_1, k) \tag{78}$$

by which we will identify these two sets. We will represent $D \in \mathrm{Dyn}^{dist}(R_1, V, F_1, k)$ by the Dynkin diagram of F_1 in which the vertex corresponding to $\alpha_i \in F_1$ is labelled by the weight $\alpha_i(D) > 0$ (or simply by the list of values $(\alpha_1(D), \ldots, \alpha_n(D))$).

Given $k \in \mathcal{K}$ let $W_0\backslash\mathrm{Res}_k^{lin}(R_1)$ be the set of orbits of generic linear residual points $W_0\xi$ such that $k \in \mathcal{K}_\xi^{reg}$. We will also describe in this section the fibers of the evaluation map

$$\mathrm{ev}_k : W_0\backslash\mathrm{Res}_k^{lin}(R_1) \to \mathrm{Dyn}^{dist}(R_1, V, F_1, k) \tag{79}$$

$$W_0\xi \to D = \xi(k)_+ \tag{80}$$

where $\xi(k)_+ \in W_0\xi(k)$ is the unique F_1-dominant element in the orbit $W_0\xi(k)$.

If $D \in \mathrm{Dyn}^{dist}(R_1, V, F_1, k)$ and $\lambda > 0$ then $\lambda D \in \mathrm{Dyn}^{dist}(R_1, V, F_1, \lambda k)$ and $-w_0(D) = D$ (using [O1, Theorem A.14(i)]). This gives canonical identifications

$$\mathrm{Dyn}^{dist}(R_1, V, F_1, \lambda k) = |\lambda|\mathrm{Dyn}^{dist}(R_1, V, F_1, k) \tag{81}$$

for all $\lambda \in \mathbb{R}^\times$. Since the generic linear residual points depend linearly on k this remark implies that we only need to describe the set $\mathrm{Dyn}^{dist}(R_1, V, F_1, k)$ and the fibres of $\widetilde{\mathrm{ev}}_k$ on all lines in the parameter space.

If $k_\alpha = 2$ for all $\alpha \in R_1$ then the set $\mathrm{Dyn}^{dist}(R_1, V, F_1, k)$ is the usual set of distinguished Dynkin diagrams, corresponding to the set of distinguished unipotent orbits of $\mathfrak{g}_\mathbb{C}(R_1)$ via the Bala-Carter theorem. For classical root systems it is known how to generalize combinatorially the set of (distinguished) unipotent classes and the Bala-Carter bijection to the set of k-weighted Dynkin diagrams [Slo2]. As this is a very useful description we will give these generalized Bala-Carter maps as well.

Consider the "degenerate" generic central character map $gcc^{\mathbf{H}}$, which is the map

$$gcc^{\mathbf{H}} : \Delta^{\mathbf{H}}(R_1, V, F_1, k) \to W_0\backslash\mathrm{Res}_k^{lin}(R_1) \tag{82}$$

corresponding to the restriction of gcc to the set $\Delta^s(\mathcal{R}, q)$ (with $s \in T_u$ a W_0-invariant element) via the canonical bijections of Corollary 2.31 and Proposition 2.56). In the next section we will prove that for all irreducible non-simply laced root systems the map $gcc^{\mathbf{H}}$ maps $\Delta^{\mathbf{H}}_{W_0D}(R_1, V, F_1, k)$ bijectively onto the fiber $\mathrm{ev}_k^{-1}(D)$ where ev_k is the evaluation map of (79) for R_1) with one remarkable exception: in the case F_4 it turns out that one has to count every occurrence of the unique singular generic linear residual orbit "f_8" with multiplicity 2. In other words, in the notation of Corollary 5.11, the multiplicities $M_{W_0r\times U}$ are always 1 for orbits W_0r of positive generic residual point, except for the unique singular one (called f_8) of F_4, in which case the multiplicity is always 2 (these results will be shown in the next section).

It is interesting in addition that this bijection also holds for type D_n after we make a small adaptation in order to see type D_n as a specialization of type B_n. The proofs of these facts do not depend on the classical Kazhdan-Lusztig classification. The only point where one needs to resort to nontrivial computations is in the verification of the fact that the multiplicity of f_8 is always 2. This follows from results by Reeder [R1]. Since our parametrization clearly also holds for type A_n it follows that the deformation method gives the classification of the discrete series in all cases except for types $E_{6,7,8}$ (in which cases the Kazdan-Lusztig classification is available of course).

In the "classical situation" $k_\alpha = 2$ for all $\alpha \in R_1$ one associates a set of Springer representations $\Sigma_{u(D)}$ of W_0 to the distinguished unipotent orbit $u = u(D)$ of $G_\mathbb{C}^{ad}(R_1)$ associated with D. The Kazhdan-Lusztig parametrization says that the set $\Delta^{\mathbf{H}}_{W_0D}(R_1, V, F_1, k_\alpha = x)$ (equal parameters with $x > 0$) is in canonical bijection with the set $\Sigma_{u(D)}$.

For classical root systems [Slo2] explained how to generalize combinatorially the set of "k-unipotent" elements $u(D)$ associated to $D \in \mathrm{Dyn}^{dist}(R_1, V, F_1, k)$ and the set of corresponding "k-Springer representations" $\Sigma_{u(D)}(k)$ of W_0. This makes it possible to recast the above parametrizations in the form of a generalized Kazhdan-Lusztig correspondence between the set $\Delta^{\mathbf{H}}_{W_0 D}(R_1, V, F_1, k_\alpha = x)$ and the sets of k-Springer representations $\Sigma_{u(D)}(k)$ on a combinatorial level for arbitrary k. Our result thus establishes this aspect of the conjectures by Slooten ([Slo2]).

We will include the generalized Kazhdan-Lusztig parameters for the classical root systems, and describe their relation with the alternative parametrization (82).

6.1. **The case $R_1 = A_n,\ n \geq 1$.** In this case $\mathcal{K} \approx \mathbb{R}$. Choose the bases of simple roots $F_1 = \{e_1 - e_2, \ldots, e_{n-1} - e_n\}$ for R_1, and define $\xi : \mathcal{K} \to V$ by the equations $\alpha(\xi(k)) = k$ for all $\alpha \in F_1$. Then $W_0 \backslash \mathrm{Res}^{lin}(R_1) = \{W_0 \xi\}$. The set $\mathcal{K}^{reg}_\xi$ is equal to $\mathcal{K}^{reg}_\xi = \mathcal{K} \backslash \{0\}$. For all $k \in \mathcal{K}^{reg}_\xi$ we have $\mathrm{Dyn}^{dist}(R_1, V, F_1, k) = \{D(k)\}$ with $D(k) = (|k|, \ldots, |k|)$. We have $\mathrm{ev}_k^{-1}(D(k)) = \{W_0 \xi\}$.

6.2. **The case $R_1 = B_n,\ n \geq 2$.** The results in this subsection are due to Slooten [Slo2]. Put $R_1 = \{\pm e_i \pm e_j \mid 1 \leq i \neq j \leq n\} \cup \{\pm e_i \mid 1 \leq i \leq n\}$. Choose as a basis $F_1 = \{e_1 - e_2, \ldots, e_{n-1} - e_n, e_n\}$. We put $k(e_i \pm e_j) = k_1 \in \mathbb{R}$ and $k(e_i) = k_2 \in \mathbb{R}$ and in this way make the identification $\mathcal{K} = \mathbb{R}^2$. If $k_1 \neq 0$ then we put $m \in \mathbb{R}$ by $m = k_2/k_1$.

We first describe the generic linear residual points. Given a partition $\lambda \in \mathcal{P}(n)$ (i.e. a partition $\lambda \vdash n$) we define a $\mathcal{K}$-valued point ξ_λ as follows. Given a box b of λ let $i(b)$ be its row number and $j(b)$ its column number. We define the content $c(b)$ of the box b by $c(b) = j(b) - i(b)$. We call the tableau of shape λ in which the boxes $b \in \lambda$ are filled with the expression $c(b)k_1 + k_2$ the *generic k-shifted tableau* of λ, denoted by $T(\lambda, k)$. We order the boxes of $T(\lambda, k)$ in the standard way by reading the tableau from left to right and from top to bottom. Then we define ξ_λ as the $\mathcal{K}$-valued point of V such that the i-th coordinate $e_i(\xi)$ is equal to the filling $c(b_i)k_1 + k_2$ of the i-th box of $T(\lambda, k)$.

Theorem 6.2. *We have a bijection*

$$\begin{aligned} \Lambda : \mathcal{P}(n) &\to W_0 \backslash \mathrm{Res}^{lin}(R_1) \\ \lambda &\to W_0 \xi_\lambda \end{aligned}$$

The set $\mathcal{K}^{reg}_\lambda$ of regular parameters for ξ_λ is of the form

$$\mathcal{K}^{reg}_\lambda = \mathcal{K} \backslash \bigcup_{m \in M^{sing}_\lambda} L_m \tag{83}$$

where where $L_m = \{(k_1, k_2) \mid k_2 = m k_1\} \subset \mathcal{K}$ and where M^{sing}_λ is a set of half-integral ratio's $m \in \mathbb{Z}/2$ which are called singular with respect to λ and which will be described in Proposition 6.4 below. We first define for $m \in \mathbb{Z}/2$ the m-shifted content tableau $T_m(\lambda)$ of λ as follows. The tableau $T_m(\lambda)$ has shape λ and the box b of $T_m(\lambda)$ is filled with the value $|c(b) + m|$ (i.e. the absolute value of the filling of the same box in $T(\lambda, (1, m))$. The following notion plays an important role:

Definition 6.3. *Let $\lambda \vdash n$ and $m \in \mathbb{Z}/2$. The list of extremities of $T_m(\lambda)$ is the weakly increasing list consisting of the following numbers. If $m \in \mathbb{Z}$ (resp. $m \in \mathbb{Z} + 1/2$) then the extremities are the fillings of the boxes of $T_m(\lambda)$ at the end of*

a row of $T_m(\lambda)$ which are on or above the 0 *diagonal (resp. the upper* 1/2*-diagonal) and the boxes at the bottom of a column of $T_m(\lambda)$ which are on or below the zero diagonal (resp. the lower* 1/2 *diagonal). Here we agree to count* 0 *twice if* 0 *is both at the the end of a row and of a column.*

Proposition 6.4. *We have $m \in M_\lambda^{reg}$ (the complement of M_λ^{sing}, i.e. the values $m \in \mathbb{R}$ such that $\xi_\lambda(k_1, mk_1)$ is residual if $k_1 \neq 0$) if and only if $m \notin \mathbb{Z}/2$ or $m \in \mathbb{Z}/2$ and the extremities of $T_m(\lambda)$ are all distinct. If $m < 1-n$ or of $m > n-1$ then m is regular with respect to any partition $\lambda \vdash n$.*

Corollary 6.5. *We have*

$$\mathcal{K}^{reg} = \mathcal{K} \backslash \bigcup_m L_m \tag{84}$$

where m runs over the half-integral values satisfying $1-n \leq m \leq n-1$. In particular, if $k \notin L_m$ for all half-integral m satisfying $1-n \leq m \leq n-1$ the evaluation map

$$\mathrm{ev}_k : W_0 \backslash \mathrm{Res}^{lin}(R_1) \to \mathrm{Dyn}^{dist}(R_1, V, F_1, k) \tag{85}$$

is bijective.

Let $m \in \mathbb{Z}/2$ and $\lambda \vdash n$. Suppose that $m \notin M_\lambda^{sing}$ (in other words $\xi_\lambda(k_1, mk_1) \in \mathrm{Res}^{lin}(R_1, V, F_1, (k_1, mk_1))$ if $k_1 \neq 0$). Since W_0 contains sign changes and permutations the corresponding element $D(k) \in \mathrm{Dyn}^{dist}(R_1, V, F_1, (k_1, mk_1))$ has coordinates which are all of the form $p|k_1|$ with $p \geq 0$ and $p \in m + \mathbb{Z}$. Conversely, any point $D(k) \in \mathrm{Dyn}^{dist}(R_1, V, F_1, k)$ is of this form. In order to see this we recall the following result (see [HO1], [Slo2]):

Proposition 6.6. *Let $m \in \mathbb{Z}/2$ and let $k = (k_1, mk_1)$ with $k_1 \neq 0$. Let $D \in \mathbb{R}^n$ be dominant with respect F_1. Then $D \in \mathrm{Dyn}^{dist}(R_1, V, F_1, k)$ only if all coordinates of D are of the form $p|k_1|$ with $p \geq 0$. So let us suppose that all coordinates of D are of the above mentioned form. Let $\mu_p = \mu_p(D)$ denote the multiplicity of $p|k_1|$ as a coordinate of D. We distinguish the following cases:*

1. *If $m = 0$ then $D \in \mathrm{Dyn}^{dist}(R_1, V, F_1, k)$ iff (i) $\mu_r = 1$ if r is maximal such that $\mu_r \neq 0$, (ii) $\mu_p \in \{\mu_{p+1}, \mu_{p+1} + 1\}$ for all $p > 0$, and (iii) $\mu_0 = \lfloor 1/2(\mu_1 + 1) \rfloor$.*
2. *If $m \in \mathbb{Z} \backslash \{0\}$ then $D \in \mathrm{Dyn}^{dist}(R_1, V, F_1, k)$ iff (i) $\mu_r = 1$ if r is maximal such that $\mu_r \neq 0$, (ii) $\mu_p \in \{\mu_{p+1}, \mu_{p+1} + 1\}$ for all $p \geq |m|$, (iii) $\mu_p \in \{\mu_{p+1} - 1, \mu_{p+1}\}$ for $1 \leq p \leq |m| - 1$, and finally (iv) $\mu_0 = \lfloor \mu_1/2 \rfloor$.*
3. *If $m \in \mathbb{Z} + 1/2$ then $D \in \mathrm{Dyn}^{dist}(R_1, V, F_1, k)$ iff (i) $\mu_r = 1$ if r is maximal such that $\mu_r \neq 0$, (ii) $\mu_p \in \{\mu_{p+1}, \mu_{p+1} + 1\}$ for all $p \geq |m|$, and finally (iii) $\mu_p \in \{\mu_{p+1} - 1, \mu_{p+1}\}$ for $1/2 \leq p \leq |m| - 1$.*

Definition 6.7. *We keep the notations as in Proposition 6.6. Assume that $D \in \mathrm{Dyn}^{dist}(R_1, V, F_1, k)$. We call $p \in m + \mathbb{Z}$ a jump of D if $p \geq |m|$ and $\mu_p = \mu_{p+1} + 1$ or if $0 < p < |m|$ and $\mu_p = \mu_{p+1}$. Finally we add 0 (if $m \in \mathbb{Z}$) or $-1/2$ (if $m \in 1/2 + \mathbb{Z}$) to the list of jumps of D in order to ensure that the number of jumps of D is equal to $\lceil |m| \rceil + 2\nu$ for some $\nu \in \mathbb{Z}_{\geq 0}$ (this is always possible-see* [Slo2]*).*

Remark 6.8. *It is a simple matter to reconstruct D from its list of jumps by computing the multiplicities m_p of the entries of the form $p|k_1|$, starting from the top $m_r = 1$.*

This gives rise to a different classification of the set of k-weighted distinguished Dynkin diagrams $\mathrm{Dyn}^{dist}(R_1, V, F_1, k)$ by the introduction of a combinatorial analogue $\mathcal{U}_m(n)$ of the corresponding set of "distinguished m-unipotent classes":

Definition 6.9. *If $m \in \mathbb{Z}$ we define*

$$\mathcal{U}_m^{dist}(n) = \{u \vdash 2n + m^2 \mid l(u) \geq |m| \text{ and } u \text{ has odd, distinct parts}\} \tag{86}$$

and if $m \in 1/2 + \mathbb{Z}$ we define

$$\mathcal{U}_m^{dist}(n) = \{u \vdash 2n + m^2 - 1/4 \mid l(u) \geq \lfloor |m| \rfloor \text{ and } u \text{ has even, distinct parts}\} \tag{87}$$

Proposition 6.10. *Let $m \in \mathbb{Z}/2$ and let $u \in \mathcal{U}_m^{dist}(n)$. Let $k = (k_1, mk_1) \in L_m$ with $k_1 \neq 0$. If $m \in 1/2+\mathbb{Z}$ we add 0 as a part of u if necessary to assure that the number of parts of u is equal to $\lceil |m| \rceil + 2\nu$ for some $\nu \in \mathbb{Z}_{\geq 0}$. The list $j = j(u)$ consisting of the numbers $(u_i - 1)/2$ where u_i runs over the parts of u (ordered in ascending order) is the list of jumps of a unique distinguished k-weighted Dynkin diagram $D \in \mathrm{Dyn}^{dist}(R_1, V, F_1, k)$ (where D is of the form as described in Proposition 6.6). This sets up a bijection*

$$f_k^{BC} : \mathcal{U}_m^{dist}(n) \to \mathrm{Dyn}^{dist}(R_1, V, F_1, k) \tag{88}$$

Finally we remark that $\mathrm{Dyn}^{dist}(R_1, V, F_1, (0,0)) = \emptyset$.

This completes the classification of the set $\mathrm{Dyn}^{dist}(R_1, V, F_1, k)$ for all values of $k \in \mathcal{K}$. It remains to describe for all special values $k \in L_m \backslash \{0\}$ and all $D \in \mathrm{Dyn}^{dist}(R_1, V, F_1, k)$ the fiber $\mathrm{ev}_k^{-1}(D)$ of the evaluation map

$$\mathrm{ev}_k : W_0 \backslash \mathrm{Res}_k^{lin}(R_1) \to \mathrm{Dyn}^{dist}(R_1, V, F_1, k) \tag{89}$$

(where $W_0 \backslash \mathrm{Res}_k^{lin}(R_1)$ is the set of orbits of generic residual points which remain residual upon evaluation at k (note that this depends on $m = m(k)$ rather than k)). Equivalently, we will describe for each $D \in \mathrm{Dyn}^{dist}(R_1, V, F_1, k)$ the set

$$\mathcal{P}_m(D) := \Lambda^{-1}(\mathrm{ev}_k^{-1}(D)) \subset \mathcal{P}(n) \tag{90}$$

of all partitions λ of n such that $W_0 \xi_\lambda(k) = W_0 D$.

Definition 6.11. *Let $m \in \mathbb{Z}/2$. Given $u \in \mathcal{U}_m^{dist}(n)$ we define a 2-partition $\phi_m(u) \in \mathcal{P}(n, 2)$ as follows. First assume that m is nonnegative. Let $j = j(u)$ be the sequence of jumps of length $\lceil m \rceil + 2\nu \in \mathbb{Z}_{\geq 0}$ associated to u as in proposition 6.10. Then we define $\phi_m(u) = (\xi_m(u), \eta_m(u)) \in \mathcal{P}(2, n)$ where*

$$\xi_m(u) = (j_1, j_3, \ldots, j_{2\nu-1}, j_{2\nu+1}, j_{2\nu+2} - 1, j_{2\nu+3} - 2, \ldots, j_{2\nu+m} - (m-1)),$$
$$\eta_m(u) = (j_2 + 1, j_4 + 1, \ldots, j_{2\nu} + 1)$$

if $m \in \mathbb{Z}$ and

$$\xi_m(u) = (j_1 + \frac{1}{2}, j_3 + \frac{1}{2}, \ldots, j_{2\nu+1} + \frac{1}{2}, j_{2\nu+2} - \frac{1}{2}, j_{2\nu+3} - \frac{3}{2}, \ldots, j_{2\nu+m+\frac{1}{2}} - (m-1)),$$
$$\eta_m(u) = (j_2 + \frac{1}{2}, j_4 + \frac{1}{2}, \ldots, j_{2\nu} + \frac{1}{2})$$

if $m \in \frac{1}{2} + \mathbb{Z}$. If $m < 0$ then we define $\phi_m(u) := (\eta_{-m}(u), \xi_{-m}(u)) \in \mathcal{P}(2, n)$.

Definition 6.12. *Let $(\xi,\eta)\in\mathcal{P}(2,n)$. Recall the equivalence class of m-symbols of (ξ,η) denoted by $\bar{\Lambda}^m(\xi,\eta)$ (if $m=0$ we use the $+$-symbol) (see* [Slo2, Definition 3.6] *for the definition of these symbols). If $(\xi,\eta)\in\mathcal{P}(2,n)$ we denote by $[(\xi,\eta)]_m$ the set of $(\xi',\eta')\in\mathcal{P}(2,n)$ such that $\bar{\Lambda}^m(\xi,\eta)$ and $\bar{\Lambda}^m(\xi',\eta')$ have representatives which contain the same entries the same number of times. For $u\in\mathcal{U}_m^{dist}(n)$ we define $\Sigma_m(u)\subset\mathcal{P}(2,n)$ by $\Sigma_m(u):=[\phi_m(u)]_m$.*

Finally the following result of Slooten gives the desired parametrization of the set $\mathcal{P}_m(D)$ (and hence of the fiber $\mathrm{ev}_k^{-1}(D)$ of the evaluation map):

Theorem 6.13. *(see* [Slo2, Theorem 5.27]*) The joining map $\mathcal{J}_m$ (see* [Slo2, Definition 5.18]*) is well defined on $\Sigma_m(u)$ and this yields a bijection*

$$\mathcal{J}_m:\Sigma_m(u)\to\mathcal{P}_m(f_k^{BC}(u)) \tag{91}$$

whose inverse is given by the splitting map $\mathcal{S}_m$ (see [Slo2, Definition 5.16]*).*

Corollary 6.14. *Let $m\in\mathbb{Z}/2$, $k=(k_1,mk_1)$ with $k_1\neq 0$ and let $D\in\mathrm{Dyn}^{dist}(R_1,V,F_1,k)$. Put $u=(f_k^{BC})^{-1}(D)\in\mathcal{U}_m(n)$. We can arrange that u has $\lceil m\rceil+2\nu$ parts (with $\nu\in\mathbb{Z}_{\geq 0}$). Then*

$$|\mathcal{P}_m(D)|=\begin{cases}\binom{\lceil m\rceil+2\nu}{\nu} & \text{if } u_1\neq 0,\\ \binom{\lceil m\rceil+2\nu-1}{\nu} & \text{otherwise.}\end{cases} \tag{92}$$

6.2.1. *The case $k_1=0$.* If $k=(0,0)$ then there are no linear residual points since k is singular for all generic linear residual points.

The situation with $k=(0,k_2)$ with $k_2\neq 0$ is an important special case. Its importance stems in part from the fact that although k is highly nongeneric *it is regular for all generic linear residual points*. In fact, all generic linear residual orbits coalesce upon specialization for $k_1=0$ to the unique orbit of residual points $W_0\xi(k)$ where ξ is defined by $\xi_i(k)=k_2$ for all $i=1,\ldots,n$. In other words, we have

$$\mathrm{Res}_k^{lin}(R_1)=\mathrm{Res}^{lin}(R_1) \tag{93}$$

and (in the coordinates $e_1,\ldots,e_n$ of V)

$$\mathrm{Dyn}^{dist}(R_1,V,F_1,k)=\{(|k_2|,\ldots,|k_2|)\} \tag{94}$$

The evaluation map ev_k is the unique map from $\mathrm{Res}^{lin}(R_1)$ to $\mathrm{Dyn}^{dist}(R_1,V,F_1,k)$.

6.3. **The case $R_1=C_n,\ n\geq 3$.** Put $R_1=\{\pm e_i\pm e_j\mid 1\leq i\neq j\leq n\}\cup\{\pm 2e_i\mid 1\leq i\leq n\}$. Choose as a basis $F_1=\{e_1-e_2,\ldots,e_{n-1}-e_n,2e_n\}$. We put $k(e_i\pm e_j)=k_1\in\mathbb{R}$ and $k(2e_i)=k_2\in\mathbb{R}$ and in this way make the identification $\mathcal{K}=\mathbb{R}^2$. Clearly we have the following equality for all $k=(k_1,k_2)$:

$$\mathrm{Res}^{lin}(C_n,(k_1,k_2))=\mathrm{Res}^{lin}(B_n,(k_1,k_2/2)) \tag{95}$$

Since $W_0(B_n)=W_0(C_n)$ we see that everything reduces to the case $R_1=B_n$.

6.4. **The case $R_1=D_n,\ n\geq 4$.** We put $R_1=\{\pm e_i\pm e_j\mid 1\leq i\neq j\leq n\}$. Choose as a basis $F_1=\{e_1-e_2,\ldots,e_{n-1}-e_n,e_{n-1}+e_n\}$. The case $R_1=D_n$ can be reduced to the discussion of subsection 6.2 as well in the following way, using the Clifford theory discussion from [RR].

Let F_1^b denote the basis for B_n as in subsection 6.2. Let

$$\psi:\mathbf{H}(B_n,V,F_1^b,(k_1,k_2))\to\mathbf{H}(B_n,V,F_1^b,(k_1,-k_2)) \tag{96}$$

be the unique algebra isomorphism such that $\psi(x) = x$ for all $x \in V^* = \mathbb{R} \otimes X$, $\psi(s_{e_{i-1}-e_i}) = s_{e_{i-1}-e_i}$ (for all $i = 2, \dots, n$) and $\psi(s_{e_n}) = -s_{e_n}$ (compare with the isomorphisms ψ_s discussed in paragraph 2.1.2). Then ψ restricts to an involutive automorphism of $\mathbf{H}(B_n, V, F_1^b, (k_1, 0))$. Let $\Psi = \{1, \psi\} \approx \mathbb{Z}/2$ be the group of automorphims of $\mathbf{H}(B_n, V, F_1^b, (k_1, 0))$ generated by ψ. Then it is easy to see that

$$\mathbf{H}(D_n, V, F_1, (k_1, 0)) \simeq \mathbf{H}(B_n, V, F_1^b, (k_1, 0))^{\Psi} \tag{97}$$

(where the generator $s_{e_{n-1}+e_n}$ on the left hand side corresponds to the element $s_{e_n} s_{e_{n-1}-e_n} s_{e_n}$ on the right hand side).

Let $k = k(\pm e_i \pm e_j) \in \mathcal{K}(D_n)$. We use k as a coordinate on the line $L_0 \subset \mathcal{K}(B_n)$ by identifying k with the element $(k, 0) \in L_0$. Let us from now assume that $k \in \mathcal{K}^{reg}(D_n) = \mathcal{K}(D_n)\backslash\{0\}$ (and in the context of $R_1 = B_n$ we identify k with $(k, 0) \in L_0$). We have $W_0(B_n) = W_0(D_n) \rtimes \Gamma$ where $\Gamma = \{e, \gamma\} \approx \mathbb{Z}/2$ and γ is the diagram automorphism that exchanges $e_{n-1} - e_n$ and $e_{n_1} + e_n$. Hence the center equals (see Corollary 2.10):

$$\mathbf{Z}(B_n, F_1^b, (k, 0)) = \mathbf{Z}(D_n, F_1, k)^{\Gamma} \tag{98}$$

It is easy to see that for every $u \in \mathcal{U}_0^{dist}(n)$ (defined as in subsection 6.2) the orbit $W_0(B_n) f_k^{BC}(u) \in W_0(B_n)\backslash \mathrm{Res}(B_n, k)$ is in fact a single $W_0(D_n)$-orbit of residual points for $R_1 = D_n$. It follows that

$$f_k^{BC} : \mathcal{U}_0^{dist}(n) \to \mathrm{Dyn}^{dist}(D_n, F_1, k) \tag{99}$$

is a bijection.

Observe that we have (using the notation of Theorem 6.2) the following relation:

$$W_0 \xi_{\lambda'}(k_1, -k_2) = W_0 \xi_{\lambda}(k_1, k_2) \tag{100}$$

where $\lambda \to \lambda'$ is the conjugation involution of $\mathcal{P}(n)$. Thus the set $W_0(B_n)\backslash \mathrm{Res}_0^{lin}(B_n)$ of orbits of generic residual B_n-points which remain residual if we restrict (k_1, k_2) to a (nonzero) element $(k, 0) \in L_0$ admits an involution ι given (via Λ) by the conjugation involution. By Proposition 6.4 this involution acts *in a fixed point free manner* on $W_0\backslash \mathrm{Res}_0^{lin}(B_n)$. The involution is clearly compatible with the evaluation map ev_0. It follows from (100) that for all $\delta \in \Delta^{\mathbf{H}}(B_n, V, F_1^b, (k, 0))$ we have

$$gcc(\delta \circ \psi) = \iota(gcc(\delta)) \tag{101}$$

Accordingly we define

$$W_0(D_n)\backslash \mathrm{Res}^{lin}(D_n)_k^{\sharp} := W_0(B_n)\backslash \mathrm{Res}_0^{lin}(B_n)/\{e, \iota\} \tag{102}$$

and we have a corresponding evaluation map

$$W_0(D_n)\backslash \mathrm{Res}^{lin}(D_n)_k^{\sharp} \to \mathrm{Dyn}^{dist}(D_n, F_1, k) \tag{103}$$

Remark 6.15. *The relation with the usual Kazhdan-Lusztig parameters for D_n is as follows. For all $u \in \mathcal{U}_0^{dist}(n)$ the involution ι acts without fixed points on the set $\Sigma_0(u)$ by:*

$$\begin{aligned} \iota : \Sigma_0(u) &\to \Sigma_0(u) \\ (\xi, \eta) &\to (\eta, \xi) \end{aligned}$$

The set $\Sigma^{D_n}(u)$ of Springer representations of $W_0(D_n)$ associated with u is the set of $\{1, \iota\}$-orbits in $\Sigma_0(u)$. In particular, for all $D \in \mathrm{Dyn}^{dist}(D_n, F_1, k)$ we have

TABLE 1. F_4: Generic linear residual orbits

Orbits $f = W_0\xi$	ξ
f_1	$\xi_1 = (k_1, k_1, k_2, k_2)$
f_2	$\xi_2 = (k_1, k_1, k_2 - k_1, k_2)$
f_3	$\xi_3 = (k_1, k_1, k_2 - k_1, k_1)$
f_4	$\xi_4 = (k_1, k_1, k_2 - 2k_1, k_2)$
f_5	$\xi_5 = (k_1, k_1, k_2 - 2k_1, 2k_1)$
f_6	$\xi_6 = (k_1, k_1, k_2 - 2k_1, k_1) \neq 0$
f_7	$\xi_7 = (k_1, k_1, k_2 - 2k_1, -2k_2)$
f_8	$\xi_8 = (0, k_1, 0, k_2 - k_1)$

a natural bijection between the fiber $(\mathrm{ev}_k^\sharp)^{-1}(D)$ *and the set of classical Kazhdan-Lusztig parameters* $\Sigma^{D_n}(u)$ *associated to* $u = u(D)$*.*

6.5. **The case** $R_1 = E_n,\ n = 6, 7, 8$**.** In the simply laced cases we can classify the generic linear residual orbits with the weighted Dynkin diagrams for the distinguished nilpotent orbits (see [O1, Proposition B.1(i)]). Since the weighted Dynkin diagrams characterize the nilpotent orbits completely by the Bala-Carter theorem (see [Car]) we obtain for all $k \neq 0$ a bijection

$$f_k^{BC} : \mathcal{U}^{dist}(R_1) \to \mathrm{Dyn}^{dist}(R_1, V, F_1, k) \tag{104}$$

where $\mathcal{U}^{dist}(R_1)$ denotes the set of distinguished nilpotent orbits of the simple complex Lie algebra with root system R_1. It is well known that the values of the roots on the generic linear residual points are integral linear combinations of the $k(\alpha)$ (corresponding to the fact that the roots take even values on the distinguished weighted Dynkin diagrams). We refer to [Car, pages 176-177] for the tables of the distinguished weighted Dynkin diagrams.

6.6. **The case** $R_1 = F_4$**.** Let $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ be a basis of simple roots of R_1 such that α_1 and α_2 are long, α_3 and α_4 are short, and $\alpha_2(\alpha_3^\vee) = -2$.

The set $W_0\backslash\mathrm{Res}^{lin}(F_4)$ was completely classified in [HO1, Table 4.10], but unfortunately this table contains an error (the coordinates of f_7 are incorrect). We therefore include the corrected table (see Table 1) below. There are eight orbits of generic linear residual points for F_4, numbered $f_1, \dots, f_8$. The orbits are generically regular with respect to the W_0-action, except for f_8 which generically has an isotropy group of type $A_1 \times A_1$. In the table below we have specified for each generic linear residual orbit $f_n = W_0\xi_n$ a generic linear residual point ξ_n by means of the vector of values $(\alpha_1(\xi_n), \dots, \alpha_4(\xi_n))$. Here $k = (k_1, k_2)$ where k_1 is the parameter of the long roots. We list in Table 3 the non-generic values of k, together with the set $\mathrm{Dyn}^{dist}(k) := \mathrm{Dyn}^{dist}(R_1, V, F_1, k)$ of k-weighted Dynkin diagrams and for each $D \in \mathrm{Dyn}^{dist}(k)$ the inverse image $\mathrm{ev}_k^{-1}(D)$ of the map

$$\mathrm{ev}_k : W_0\backslash\mathrm{Res}_k^{lin} \to \mathrm{Dyn}^{dist}(k) \tag{105}$$

Remark 6.16. *In Table 3 we assume that* $x > 0$*. Not all special parameters are listed in table 3 but all other special values can be obtained from the listed ones by applying the following symmetries. First of all we have* $f_i(k_1, k_2) = f_i(-k_1, -k_2)$ *(since*

TABLE 2. F_4: Regular parameters

Orbit	$\mathcal{K}_\xi^{reg}$
f_1	$(2k_1+3k_2)(3k_1+4k_2)(3k_1+5k_2)(5k_1+6k_2) \neq 0$
f_2	$(k_1^2-(6k_2)^2)k_2 \neq 0$
f_3	$(3k_1+2k_2)(k_1+3k_2)(2k_1+3k_2)(3k_1+4k_2) \neq 0$
f_4	$(2k_1-3k_2)(3k_1-4k_2)(3k_1-5k_2)(5k_1-6k_2) \neq 0$
f_5	$((3k_1)^2-(2k_2)^2)(k_1^2-(3k_2)^2) \neq 0$
f_6	$(3k_1-2k_2)(k_1-3k_2)(2k_1-3k_2)(3k_1-4k_2) \neq 0$
f_7	$((3k_1)^2-k_2^2)k_1 \neq 0$
f_8	$k_1k_2 \neq 0$

$-\mathrm{id} \in W_0$*) and* $f_i(k_1,k_2) = f_{\theta(i)}(k_1,-k_2) = f_{\theta(i)}(-k_1,k_2)$ *with* $\theta = (14)(36)$*. With these transformations we can reach all quadrants of* $\mathcal{K}$ *from the positive quadrant. In addition we have used the following symmetry (arising from interchanging the long and short roots) to reduce the length of Table 3: Let* $\Psi(a,b,c,d) = (2d,2c,b,a)$*. Then we can define* $D_i(2k_2,k_1)$ *by* $D_i(2k_2,k_1) = \Psi(D_i(k_1,k_2))$*. The map* Ψ *acts as follows on the set of generic linear residual orbits:* $\Psi(f_i(k_1,k_2)) = f_{\sigma(i)}(2k_2,k_1)$ *where* σ *is the transposition* (27)*. Observe that* $\Psi^2(a,b,c,d) = (2a,2b,2c,2d)$*, thus* Ψ^2 *corresponds to replacing* x *by* $2x$*.*

6.7. **The case** $R_1 = G_2$**.** See [HO1, Proposition 4.15]. There are three orbits of generic linear residual points $W_0\xi_1$, $W_0\xi_2$ and $W_0\xi_3$. which we will refer to as g_1, g_2, and g_3. Let α_1 be the simple long root and α_2 the simple short root. Let $k=(k_1,k_2)$ with k_1 the parameter of the long root. The following table lists the $g_i = W_0\xi_i$ and the set $\mathcal{K}_i^{reg}$ where $W_0\xi_i$ remains residual upon specialization. We use similar conventions as in the case F_4. We list in Table 5 the non-generic values of k, together with the set $\mathrm{Dyn}^{dist}(k)$ of k-weighted Dynkin diagrams and for each $D \in \mathrm{Dyn}^{dist}(k)$ the inverse image $\mathrm{ev}_k^{-1}(D)$ of the map

$$\mathrm{ev}_k : W_0\backslash\mathrm{Res}_k^{lin} \to \mathrm{Dyn}^{dist}(k) \tag{106}$$

Remark 6.17. *In Table 5 we assume that* $x>0$*. Not all special parameters are listed in table 5 but all other special values can be obtained from the listed ones by applying the following symmetries. First of all we have* $g_i(k_1,k_2) = g_i(-k_1,-k_2)$ *(since* $-\mathrm{id} \in W_0$*) and* $g_i(k_1,k_2) = g_{\theta(i)}(k_1,-k_2) = g_{\theta(i)}(-k_1,k_2)$ *with* $\theta = (12)$*. With these transformations we can reach all quadrants of* $\mathcal{K}$ *from the positive quadrant. In addition we have used the following symmetry (arising from interchanging the long and short roots) to reduce the length of Table 5: Let* $\Psi(a,b) = (3b,a)$*. Then we can define* $D_i(3k_2,k_1)$ *by* $D_i(3k_2,k_1) = \Psi(D_i(k_1,k_2))$*. The map* Ψ *acts as follows on the set of generic linear residual orbits:* $\Psi(f_i(k_1,k_2)) = f_i(3k_2,k_1)$*. Observe that* $\Psi^2(a,b) = (3a,3b)$*, thus* Ψ^2 *corresponds to replacing* x *by* $3x$*.*

7. THE CLASSIFICATION OF THE DISCRETE SERIES OF $\mathbf{H}$

We formulate the main theorem of this paper.

TABLE 3. k-weighted Dynkin diagrams and confluence data for F_4

$k = (k_1, k_2)$	$D \in \mathrm{Dyn}^{dist}(k)$	$\mathrm{ev}_k^{-1}(D)$
$(0, x)$	$D_1 = (0, 0, x, x)$	f_1, f_2, f_4
	$D_2 = (0, 0, x, 0)$	f_3, f_5, f_6
(x, x)	$D_1 = (x, x, x, x)$	f_1
	$D_2 = (x, x, 0, x)$	f_2, f_3
	$D_3 = (0, x, 0, x)$	f_5, f_7
	$D_4 = (0, x, 0, 0)$	f_4, f_6, f_8
$(x, 2x)$	$D_1 = (x, x, 2x, 2x)$	f_1
	$D_2 = (x, x, x, 2x)$	f_2
	$D_3 = (x, x, x, x)$	f_3
	$D_4 = (x, x, 0, 2x)$	f_4, f_5
	$D_5 = (x, x, 0, x)$	f_6, f_7
	$D_6 = (0, x, 0, x)$	f_8
$(x, 3x)$	$D_1 = (x, x, 3x, 3x)$	f_1
	$D_2 = (x, x, 2x, 3x)$	f_2
	$D_3 = (x, x, x, 3x)$	f_4
	$D_4 = (x, x, 2x, x)$	f_3
	$D_5 = (x, x, x, 2x)$	f_5
	$D_6 = (x, x, x, x)$	f_6
	$D_7 = (0, x, 0, 2x)$	f_8
$(2x, 3x)$	$D_1 = (2x, 2x, 3x, 3x)$	f_1
	$D_2 = (2x, 2x, x, 3x)$	f_2
	$D_3 = (2x, 2x, x, 2x)$	f_3
	$D_4 = (2x, 0, x, 2x)$	f_4, f_7
	$D_5 = (0, 2x, 0, x)$	f_8
$(3x, 2x)$	$D_1 = (3x, 3x, 2x, 2x)$	f_1
	$D_2 = (3x, x, x, 2x)$	f_3
	$D_3 = (3x, x, x, x)$	f_2
	$D_4 = (2x, x, x, 2x)$	f_7
	$D_5 = (2x, x, x, x)$	f_5
	$D_6 = (0, x, x, 0)$	f_8
$(5x, 3x)$	$D_1 = (5x, 5x, 3x, 3x)$	f_1
	$D_2 = (5x, x, 2x, 3x)$	f_3
	$D_3 = (5x, x, 2x, x)$	f_2
	$D_4 = (4x, x, 2x, 3x)$	f_7
	$D_5 = (4x, x, 2x, x)$	f_5
	$D_6 = (x, x, x, x)$	f_6
	$D_7 = (0, x, 2x, 0)$	f_8

Theorem 7.1. *Let $R_1 \subset V^*$ be a non-simply laced irreducible root system or $R_1 = A_n$. Let F_1 be a basis of simple roots, and let $k \in \mathcal{K}$. We denote by $\Delta^{\mathbf{H}}(R_1, V, F_1, k)$ the set of irreducible discrete series characters of $\mathbf{H}(R_1, V, F_1, k)$. The generic central*

TABLE 4. Generic linear residual orbits for G_2

Type	ξ	$\mathcal{K}_{\xi}^{reg}$
g_1	$\xi_1 = (k_1, k_2)$	$(k_1 + 2k_2)(2k_1 + 3k_2) \neq 0$
g_2	$\xi_2 = (k_1, k_2 - k_1)$	$(k_1 - 2k_2)(2k_1 - 3k_2) \neq 0$
g_3	$\xi_3 = (k_1, 1/2(k_2 - k_1))$	$k_1 k_2 \neq 0$

TABLE 5. k-weighted Dynkin diagrams and confluence for G_2

$k = (k_1, k_2)$	$D \in \mathrm{Dyn}^{dist}(k)$	$\mathrm{ev}_k^{-1}(D)$
$(0, x)$	$D_1 = (0, x)$	g_1, g_2
(x, x)	$D_1 = (x, x)$	g_1
	$D_2 = (x, 0)$	g_2, g_3
$(2x, x)$	$D_1 = (2x, x)$	g_1
	$D_2 = (\frac{1}{2}x, \frac{1}{2}x)$	g_3

character map induces a bijection

$$gcc_k^{\mathbf{H}} : \Delta^{\mathbf{H}}(R_1, V, F_1, k) \xrightarrow{\simeq} W_0\backslash\mathrm{Res}_k^{lin}(R_1) \tag{107}$$

which is compatible with the central character map in the sense that $\mathrm{ev}_k(gcc_k^{\mathbf{H}}(\delta)) = cc(\delta)$ *for all* $k \in \mathcal{K}$ *and for all* $\delta \in \Delta^{\mathbf{H}}(R_1, V, F_1, k)$, except *when* $R_1 = F_4$ *and* $k \in \mathcal{K}_{f_8}^{reg}$*, in which case there are exactly two elements* $\delta_{f_8'}, \delta_{f_8''} \in \Delta^{\mathbf{H}}(R_1, V, F_1, k)$ *with generic central character* f_8*. This statement is also true for* $R_1 = D_n$ *(with* $n \geq 4$*) if we replace* $W_0(D_n)\backslash\mathrm{Res}_k^{lin}(D_n)$ *by* $W_0(D_n)\backslash\mathrm{Res}_k^{lin}(D_n)^\sharp$ *and* $gcc_k^{\mathbf{H}}$ *by the map* $gcc_k^{\mathbf{H},\sharp}$ *which is equal to the map* $gcc_{(k,0)}^{\mathbf{H},B_n}$ *for type* B_n*, composed with the induction map for characters of* $\mathbf{H}(D_n, V, F_1, k)$ *to* $\mathbf{H}(B_n, V, F_1^b, (k, 0))$*.*

Proof. We apply the reduction results Corollary 2.30 and Corollary 2.31 with $u = 1$. In this situation we will denote the natural map $\mathcal{Q} \to \mathcal{K}$ given by $q \to k_{u=1} = k$ by $k = 2\log(q)$.

In view of Proposition 2.56, Corollary 2.31 and Corollary 5.11 the result is equivalent to the statement that for all $W_0\xi \in W_0\backslash\mathrm{Res}_k^{lin}(R_1)$ and all components $U \in \mathcal{K}_{W_0\xi}$ we have $M_{\{W_0\exp(\xi)\}\times\exp(U)} = 1$ except when $R_1 = F_4$ and $W_0\xi = f_8$, in which case the value should be 2 (independent of the choice of U).

If $R_1 = A_n$ (with $n \geq 1$) then there is one generic residual orbit $W_0\xi$, with two components $\mathcal{K}_{W_0\xi} = \{U_+, U_-\}$. It is of course well known in this case that $M_\pm := M_{\{W_0\exp(\xi)\}\times\exp(U_\pm)} = 1$ and there are many possible proofs for this fact, but we will explain the proof that is central to the approach in this paper in order to illustrate the method in this basic case.

The multiplicities $M_\pm$ are on the one hand at least 1 (by Corollary 5.11) and on the other hand at most 1 by Corollary 5.11, Corollary 2.31, and Corollary 2.36. This proves the required equality.

If $R_1 = B_n$ (with $n \geq 2$) we argue in a similar way. By Corollary 5.11 and Corollary 6.5 we see that for all $k \in \mathcal{K}^{gen}$ the cardinality $|\Delta^{\mathbf{H}}(R_1, V, F_1, k)| \geq |\mathcal{P}(n)|$ with equality iff $M_{\{W_0\exp(\xi)\}\times\exp(U)} = 1$ for all U such that $k \in U$. On the other

hand it is well known that the set of elliptic conjugacy classes of $W_0(B_n)$ is naturally in bijection with the set $\mathcal{P}(n)$. Hence Corollary 2.31 and Corollary 2.36 show that $|\Delta^{\mathbf{H}}(R_1, V, F_1, k)| \leq |\mathcal{P}(n)|$. We conclude that $|\Delta^{\mathbf{H}}(R_1, V, F_1, k)| = |\mathcal{P}(n)|$ and thus that $M_{\{W_0 \exp(\xi)\} \times \exp(U)} = 1$ for all orbits $W_0\xi$ and all $U \in \mathcal{K}_{W_0\xi}$ such that $U \ni k$. Since k was chosen arbitrarily we see that $M_{\{W_0 \exp(\xi)\} \times \exp(U)} = 1$ for all $W_0\xi$ and $U \in \mathcal{C}_{W_0 \exp(\xi)}$, as desired.

If $R_1 = C_n$ then the result follows easily from the case $R_1 = B_n$ using that fact that $\mathbf{H}(B_n, (k_1, k_2)) \simeq \mathbf{H}(C_n, (k_1, k_2/2))$.

If $R_1 = G_2$ the argument is completely analogous to the case $R_1 = B_n$, using the results of subsection 6.7.

In the case $R_1 = F_4$ we need additional arguments. The Weyl group $W_0(F_4)$ has 9 elliptic conjugacy classes, but by Subsection 6.6 we see that there are only 8 generic linear residual points $f_1, \dots, f_8$. The points $f_1, \dots, f_7$ are (generically) regular. A generic residual orbit $W_0 \exp(\xi(k))$ carries precisely 1 irreducible discrete series character (see [Slo1, Corollary 1.2.11]), proving that the multiplicities associated to these orbits are all precisely equal to 1. Now consider f_8. By the above numerology we see that for any component U of $\mathcal{K}_{f_8}^{reg}$ the value of $M_{f_8 \times U}$ can be either 1 or 2 and in the rest of the proof we will show that it has to be always 2. From Table 2 we have $\mathcal{K}_{f_8}^{reg} = \{U_{\pm,\pm}\}$ with $U_{\epsilon_1,\epsilon_2} = \{(k_1, k_2) \mid \epsilon_i k_i > 0 (i = 1, 2)\}$. This simple structure of $\mathcal{K}_{f_8}^{reg}$ is very helpful at this point. There exist standard automorphisms (for $\epsilon_i = \pm 1$)

$$\psi_{\epsilon_1,\epsilon_2} : \mathbf{H}(R_1, V, F_1, (k_1, k_2)) \to \mathbf{H}(R_1, V, F_1, (\epsilon_1 k_1, \epsilon_2 k_2)) \tag{108}$$

such that $\psi_{\epsilon_1,\epsilon_2}(x) = x$ for all $x \in V^*$, $\psi_{\epsilon_1,\epsilon_2}(s_i) = \epsilon_1 s_i$ (for $i = 1$, 2) and $\psi_{\epsilon_1,\epsilon_2}(s_j) = \epsilon_2 s_j$ (for $j = 3$, 4). Clearly twisting by $\psi_{\epsilon_1,\epsilon_2}$ sends discrete series characters to discrete series characters and thus that the multiplicities $M_{f_8 \times U}$ are independent of U. It was shown by Mark Reeder [R1] that there exist 2 irreducible discrete series with central character $\mathrm{ev}_{(4x,x)}(f_8)$ for the (generic) parameters $(4x, x)$ (with $x > 0$). In Reeder's parametrization these characters are called $[A_1E_7(a_5), -21]$ and $[A_1E_7(a_5), -3]$. Reeder's result is based on the explicit computation of the weight diagrams of the discrete series modules (alternatively we could invoke here the standard Kazhdan-Lusztig classification for the parameters (x, x) (with $x > 0$) to arrive at the same conclusion).

Finally let us consider the case $R_1 = D_n$. Of course this simply laced case can be treated directly by the Kazhdan-Lusztig classification (see Remark 6.15) but we want to show here how to adapt the deformation method to so that the classification for $R_1 = D_n$ is also treated by an appropriate version of the generic central character map. It was shown in Subsection 6.4 that the degenerated affine Hecke algebra $\mathbf{H}(D_n, V, F_1, k)$ is the fixed point algebra of $\mathbf{H}(B_n, V, F_1, (k, 0))$ for the action of the automorphism group $\Psi \approx \mathbb{Z}/2$. From our knowledge of the case $R_1 = B_n$ we know already that the map generic central character map $gcc^{\mathbf{H},B_n}_{(k,0)}$ for type B_n yields a bijection between $\Delta^{\mathbf{H}}(B_n, V, F_1^b, (k, 0))$ and $W_0 \backslash \mathrm{Res}_0^{lin}(B_n)$. In Subsection 6.4 we have seen that twisting by ψ acts freely on the set of generic linear residual orbits $W_0 \backslash \mathrm{Res}_0^{lin}(B_n)$. It follows that twisting by ψ acts freely on $\Delta^{\mathbf{H}}(B_n, V, F_1^b, (k, 0))$ as well. Using [RR, Theorem A.6, Theorem A.13] we see that all characters in $\Delta^{\mathbf{H}}(B_n, V, F_1^b, (k, 0))$ remain irreducible when restricted to $\mathbf{H}(D_n, V, F_1, k) = \mathbf{H}(B_n, V, F_1^b, (k, 0))^{\Psi}$, that all $\delta \in \Delta^{\mathbf{H}}(D_n, V, F_1, k)$ arise in

this way and that there always exist precisely two irreducible characters δ_+, $\delta_- \in \Delta^{\mathbf{H}}(B_n, V, F_1^b, (k,0))$ restricting to δ, and these two characters are ψ-twists of each other. This proves the required result. $\square$

Let us look at an interesting special case:

Example 7.2. *We have* $\mathbf{H}(B_n, V, F_1, (0, k_2)) \simeq \mathbf{H}(A_1^n, V, F_1(A_1^n), k_2) \rtimes S_n$ *with* $F_1^A = \{e_1, \dots, e_n\}$*. Using this it is easy to see that for* $k_2 \neq 0$

$$\Delta^{\mathbf{H}}(B_n, V, F_1, (0, k_2)) = \{\delta_\pi \mid \pi \in \widehat{S_n}\} \tag{109}$$

with $\delta_\pi = \delta^{\otimes n} \otimes \pi$ *and where* δ *is the unique irreducible (one dimensional) discrete series character of* $\mathbf{H}(A_1, V(A_1), F_1(A_1), k_2)$*. If* $k_2 > 0$ *then*

$$\delta_{\pi(\lambda)}|_{W_0} = \chi(-, \lambda') \tag{110}$$

and if $k_2 < 0$ *then*

$$\delta_{\pi(\lambda)}|_{W_0} = \chi(\lambda, -) \tag{111}$$

where $\{\pi(\lambda)\}_{\lambda \in \mathcal{P}(n)}$ *denotes the usual parametrization of the irreducible characters of* S_n *by partitions of* n *(see e.g.* [Car]*), and where* $\{\chi(\tau, \sigma)\}_{(\tau,\sigma) \in \mathcal{P}(2,n)}$ *is the usual parametrization of the irreducible characters of* $W_0 = W(B_n)$ *by 2-partitions of* n*.*

On the other hand we recall from subsection 6.2.1 that $k = (0, k_2)$ *is a regular parameter for all generic linear residual orbits of* $\mathbf{H}(B_n, V, F_1, (k_1, k_2))$*. Hence the map*

$$gcc_{(0,k_2)} : \Delta^{\mathbf{H}}(B_n, V, F_1, (0, k_2)) \to W_0 \backslash \mathrm{Res}^{lin}(B_n) \tag{112}$$

is a bijection by Theorem 7.1. By continuity (see Theorem 5.7 and Definition 5.10) it follows that for all $\lambda \in \mathcal{P}(n)$ *the generic irreducible discrete series character* $\delta_{W_0\xi_\lambda \times U_{\pm\infty}}$ *whose domain of definition is the unique connected component* $U_{\pm\infty} = U_{W_0\xi_\lambda, \pm\infty}$ *of* $\mathcal{K}^{reg}_{W_0\xi_\lambda}$ *which contains* $(0, k_2)$ *for* $\pm k_2 > 0$ *restricts to an irreducible character of* S_n*, and this sets up a bijective correspondence between the set of generic linear residual orbits and the set of irreducible characters of* S_n*.*

Remark 7.3. *Unfortunately we do not know how to compute the generic central character map in this case. We conjecture that*

$$gcc_{(0,k_2)}(\delta_{\pi(\lambda)}) = W_0\xi_{\lambda'} \tag{113}$$

if $k_2 > 0$ *and*

$$gcc_{(0,k_2)}(\delta_{\pi(\lambda)}) = W_0\xi_{\lambda} \tag{114}$$

if $k_2 < 0$*.*

The following corollary of Theorem 7.1 was known for degenerate affine Hecke algebras with equal parameters by the work of Reeder [R2].

Corollary 7.4. *Let* $k \in \mathcal{K}^{reg}$ *be a regular parameter. The elliptic pairing is positive definite on* $\mathrm{Ell}(\mathbf{H}(R_1, V, F_1, k))$ *and the map*

$$\begin{aligned} \mathrm{Ell}(\mathbf{H}(R_1, V, F_1, k)) &\to \mathrm{Ell}(W_0) \\ [\pi] &\to [\pi|_{W_0}] \end{aligned}$$

yields an isometric isomorphism with respect to the elliptic pairing.

Proof. We may assume that R_1 is irreducible. If R_1 is non-simply laced we see from our results above that (since $k \in \mathcal{K}^{reg}$) the images in $\mathrm{Ell}(\mathcal{H}(R_1, V, F_1, k))$ of the irreducible characters in $\Delta^{\mathbf{H}}(R_1, V, F_1, k)$ form a linear basis of $\mathrm{Ell}(\mathcal{H}(R_1, V, F_1, k))$. We also know that these even form an orthonormal basis with respect to the elliptic pairing, hence the elliptic pairing is positive definite in this case. Using results of [OS] it follows that the limits of these characters for xk (with $x \to 0$) from an orthonormal set of elliptic characters of W_0 (actually, in order to see this using the results of [OS] we need to lift the characters to $\mathcal{H}(\mathcal{R}, q)$ using the equivalence of Corollary 2.31, then take the limit q^x with $x \to 0$ to get a set of orthonormal elliptic characters for W, and then use the formula for the elliptic paring of [OS, Theorem 3.2]). Finally we already established in the previous theorem that the cardinality of this set is equal to the dimension $\mathrm{ell}(W_0)$ of the space $\mathrm{Ell}(W_0)$. This yields the desired result for non-simply laced cases. For simply laced cases (or more generally all cases with equal parameters k (i.e. such that $k_\alpha = x$ for all $\alpha \in R_1$) the result is due to Reeder [R2] (based on the Kazhdan-Lusztig model for the characters of $\mathcal{H}(\mathcal{R}, q)$). □

It is natural to expect that the result of Corollary 7.4 holds for arbitrary k. We conjecture something stronger (see [ABP] for related conjectures):

Conjecture 7.5. *A generic family δ of irreducible discrete series characters $\delta \in \Delta^{\mathbf{H},gen}(R_1, V, F_1)$ with domain of definition $U \in \mathcal{K}^{reg}_{W_0\xi}$ say, has weakly continuous limits to the points $k \in \overline{U}$ (the closure of U). In view of the above results this would imply that the elliptic pairing is positive definite on* $\mathrm{Ell}(\mathbf{H}(R_1, V, F_1, k))$ *for all semisimple root systems R_1 and all $k \in \mathcal{K}$, and that this space is isometric to* $\mathrm{Ell}(W_0)$ *for all $k \in \mathcal{K}$.*

Remark 7.6. *Using the $gcc^{\mathbf{H}}$ invariant is not difficult to check that for all irreducible root systems R_1 the irreducible discrete series characters are stable for twisting by diagram automorphisms (a case-by-case verification).*

8. The classification of the discrete series of $\mathcal{H}$

Since a semisimple root datum is in general not isomorphic to a direct sums of irreducible root data the classification of the irreducible discrete series characters can not be reduced to the same problem for an irreducible root datum. However, we have seen (Theorem 2.6 and Theorem 2.8) how to reduce the problem to the analogous problem for cross products of semisimple degenerate affine algebras by certain groups of diagram automorphisms. In Section 7.1 we have covered the basic building blocks, the simple degenerate affine Hecke algebras.

Even though the classification problem for semisimple affine Hecke algebras can in general not be reduced to the simple cases it is instructive to give the classification in certain basic situations. This is what we seek to do in the present section. In particular we classify in this section the irreducible discrete series characters for all the irreducible non-simply laced root data and all possible positive root labels (using Theorem 2.6 and Theorem 2.8 to reduce the problem to Theorem 7.1).

Let $\mathcal{R} = (X, R_0, Y, R_0^\vee, F_0)$ be an *irreducible* root datum, and let $q \in \mathcal{Q} = \mathcal{Q}(\mathcal{R})$. Recall the maximal root datum $\mathcal{R}^{max}$ (with $X^{max} = P(R_1)$, the weight lattice of R_1, and $R_0^{max} = R_0$) with the natural isogeny $\psi : \mathcal{R} \to \mathcal{R}^{max}$ such that $\mathcal{Q}(\mathcal{R}) =$

$\mathcal{Q}(\mathcal{R}^{max})$. Let us define

$$\Gamma = Y/Q(R_1^\vee) \simeq \mathrm{Hom}(X^{max}/X, \mathbb{C}^\times) \subset T^{max} \tag{115}$$

An element $\gamma \in \Gamma$ uniquely extends to a linear character (also denoted γ) of $W^{max} = X^{max} \rtimes W_0$ which is trivial on W_0. Γ acts on the affine Hecke algebra $\mathcal{H}^{max} = \mathcal{H}(\mathcal{R}^{max}, q)$ by means of algebra isomorphisms as follows: for $w \in W^{max}$ and $\gamma \in \Gamma$ we define $\gamma(N_w) = \gamma(w)N_w$. With this action of Γ we have

$$\mathcal{H}(\mathcal{R}, q) = \mathcal{H}(\mathcal{R}^{max}, q)^\Gamma \tag{116}$$

We are interested to apply Theorem 2.6 to central characters which carry discrete series characters of $\mathcal{H}$, in other words to orbits $W_0 r \in \mathrm{Res}(\mathcal{R}, q)$ of residual points in T. We know that $r \in T$ is of the form $r = s\exp(\xi)$ with $s \in T_u$ such that

$$R_{s,1} = \{\alpha \in R_1 \mid \alpha(s) = 1\} \tag{117}$$

is of maximal rank, and ξ is a linear $(R_{s,1}, k_s)$-residual point. Let us define $W^\vee = W_0 \ltimes 2\pi i Y$, then the action groupoid of the action of W_0 on T is equivalent to the action groupoid of $W^\vee$ acting on iV. We have a splitting of the form

$$W^\vee = W^\vee(\mathcal{R}^{max}) \rtimes \Gamma \tag{118}$$

with $W^\vee(\mathcal{R}^{max}) = W(R_1^{(1)}) = W_0 \ltimes 2\pi i Q(R_1^\vee)$ on iV, and where Γ acts on $W(R_1^{(1)})$ via diagram automorphisms of $R_1^{(1)}$. Hence we may assume that $s(e) = \exp(e)$ with $e \in E(C^\vee)$, the set of extremal point of the closure of the fundamental alcove $\overline{C^\vee}$ of $W(R_1^{(1)})$. It follows that we have

$$W_{s(e)} \simeq W(R_{s(e),1}) \rtimes \Gamma_{s(e)} \tag{119}$$

with $\Gamma_{s(e)} \simeq \{\gamma \in \Gamma \mid \gamma(e) = e\}$ (compare with Definition 2.5 and Corollary 2.54).

Let $F^\vee$ be the set of simple affine roots of $R_1^{(1)}$. If $a^\vee \in F^\vee$ then there exists a unique extremal point $e(a^\vee) \in E(C^\vee)$ such that $a^\vee(e(a^\vee)) \neq 0$. This sets up a canonical bijection $F^\vee \longleftrightarrow E(C^\vee)$ which we denote by $e \to a^\vee(e)$ and $a^\vee \to e(a^\vee)$.

Let $D(a^\vee) \in V^*$ denote the gradient of $a^\vee$. By the above, if $e \neq e(a^\vee)$ then $D(a^\vee)(s(e)) = 1$. Hence if $D(a^\vee)$ can be written as $D(a^\vee) = 2\beta$ with $\beta \in R_0$ then $\beta(s(e)) = \pm 1$ for all extremal points $e \in \overline{C^\vee}$ with $e \neq e(a^\vee)$. In this situation the value $\beta(s(e)) \in \{\pm 1\}$ is independent of the choice of $e \neq e(a^\vee)$ (namely, it equals -1 iff $\{a^\vee\} = F^\vee \backslash F_1$). Thus the following definition makes sense (in view of (26)):

Definition 8.1. *We define the spectral diagram Σ associated with $(\mathcal{R}, q)$ as the affine Dynkin diagram of $W^\vee$ associated with the basis $F^\vee$ of $R_1^{(1)}$, where we give all the vertices $a^\vee \in F^\vee$ of Σ a weight $k_{a^\vee}$ defined as follows. We define $k_{a^\vee} = k_{s,D(a^\vee)}$ (as in (26)) where $s = s(e)$ for $e \in E(C^\vee)\backslash\{e(a^\vee)\}$ (an arbitrary choice). Note that Σ (labelled with these weights) is invariant for the natural action of Γ on $F^\vee$. We include the action of Γ on the diagram and the marking of the special vertex (extending the diagram of R_1) in the spectral diagram.*

Example 8.2. *If $\mathcal{R} = \mathcal{R}^{max}$ we have $\Gamma = 1$. These cases are referred to as $R_1^{(1)}$.*

Example 8.3. *It is possible that the generic affine Hecke algebra of a root datum is a specialization of the generic affine Hecke algebra of another root datum. For example, $\mathcal{H}(C_n, P(C_n), B_n, Q(B_n), F_0(C_n))$ is isomorphic to the specialization $v_{\beta^\vee} = 1$ in the generic algebra of the type $\mathcal{H}(B_n, Q(B_n), C_n, P(C_n), F_0(B_n))$ where $\beta \in R_0 = B_n$ is*

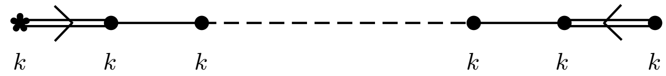

FIGURE 1. Spectral diagram of the Iwahori Hecke algebra of $SO_{2n+1}(F)$

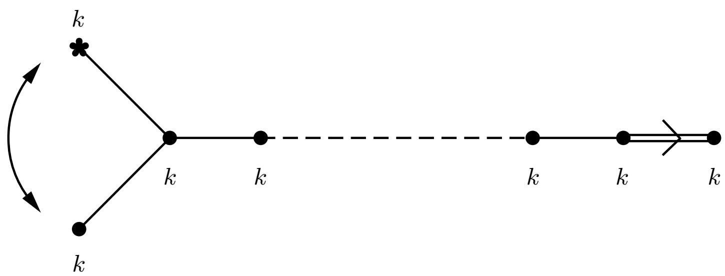

FIGURE 2. Spectral diagram of the Iwahori Hecke algebra of $Sp_{2n}(F)$

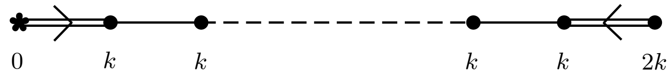

FIGURE 3. Equivalent $C_n^{(1)}$-type spectral diagram of the Iwahori Hecke algebra of $Sp_{2n}(F)$

such that $2\beta \in R_1$. This is compatible with the previous remark in the sense that both these cases are referred to as $C_n^{(1)}$. A basic example in this class is the Iwahori Hecke algebra of the Chevalley group of type $G = SO_{2n+1}(F)$, with $q^2 = |\log(\mathcal{O}/\mathcal{P})|$, the cardinality of the residue field. See Figure 1 (with $k = 2\log(q)$).

Example 8.4. *The Iwahori Hecke algebra of the simply connected group $Sp_{2n}(F)$ (where we put $q^2 = |\log(\mathcal{O}/\mathcal{P})|$) has the spectral diagram displayed in Figure 2 (where $k = 2\log(q)$). It corresponds to the case $R_0 = B_n$, $X = Q(R_0)$ and therefore it is obviously also a specialization of $C_n^{(1)}$ (namely this case corresponds to the specialization $v_{\alpha^\vee} = 1$ for $\alpha = 2\beta$ with $\beta \in R_0$). Indeed, the spectral diagram of Figure 2 is equivalent to the diagram of type $C_n^{(1)}$ displayed in Figure 3.*

Example 8.5. *More generally, let $\mathcal{R}$ be of type $C_n^{(1)}$. Let $R_0 = \{\pm e_i, \pm e_i \pm e_j\}$ and put $X = Q(R_0)$. Choose $F_0 = \{e_1 - e_2, \dots, e_{n-1} - e_n, e_n\}$ and put $q_1 = q(s_{x_i - x_{i+1}})$, $q_2 = q(s_{2x_n})$ and $q_0 = q(s_{1-2x_1})$. Put $k = 2\log(q_1)$ and define $m_\pm$ by $m_\pm k = \pm\log(q_0) + \log(q_2)$. The corresponding spectral diagram is displayed in Figure 4. We refer to* [L2], [Blo] *for explicit examples of such affine Hecke algebras as convolution algebras in the representation theory of p-adic groups.*

Definition 8.6. *For each element $e \in E(C^\vee)$ we associate the semisimple root system $R_{s(e),1}$ with basis $F_{s(e),1}$ (as in Definition 2.5). Then $D(F^\vee \backslash \{a^\vee(e)\})$ is a basis for $R_{s(e),1}$. Let $k_e \in \mathcal{K}(R_{s(e),1})$ denote the unique parameter function on $R_{s(e),1}$ which corresponds to the set of weights of Σ restricted to $F^\vee \backslash \{a^\vee(e)\}$. Then we*

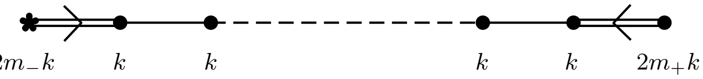

FIGURE 4. Spectral diagram for the general $C_n^{(1)}$-case

associate to e *the algebra*

$$\mathbf{H}_e := \mathbf{H}(R_{s(e),1}, V, F_{s(e),1}, k_e) \rtimes \Gamma_{s(e)} \tag{120}$$

We denote by $\Delta(\mathbf{H}_e)$ *the set of irreducible discrete series characters of* $\mathbf{H}_e$ *(in the sense as explained in the text following Corollary 2.27).*

Let us finally formulate our classification theorem:

Theorem 8.7. *Let* $\mathcal{R} = (X, R_0, Y, R_0^\vee, F_0)$ *be a root datum with* R_0 *irreducible, and let* $q \in \mathcal{Q}$*. Let* $\Delta(\mathcal{R}, q)$ *be the set of irreducible discrete series characters of the Hecke algebra* $\mathcal{H}(\mathcal{R}, q)$ *as usual. There exists a natural bijection*

$$\Delta(\mathcal{R}, q) \simeq \coprod_{e \in \Gamma\backslash E(C^\vee)} \Delta^{s(e)}(\mathcal{R}, q) \tag{121}$$

where the disjoint union is taken over a set of representatives for the Γ*-action on* $E(C^\vee)$*. For each* $e \in E(C^\vee)$ *there is a natural bijection*

$$\Delta^{s(e)}(\mathcal{R}, q) \simeq \Delta(\mathbf{H}_e) \tag{122}$$

(where the right hand side denotes the set of irreducible discrete series characters of $\mathbf{H}_e$*). In particular, if* $\Gamma_{s(e)} = 1$ *we have*

$$\Delta^{s(e)}(\mathcal{R}, q) \simeq \Delta^{\mathbf{H}}(R_{s(e),1}, V, F_{s(e),1}, k_e) \tag{123}$$

(which is completely described by Theorem 7.1). If $\delta^{\mathbf{H}} \in \Delta(\mathbf{H}_e)$ *then its restriction to* $\mathbf{H}(R_{s(e),1}, V, F_{s(e),1}, k_e)$ *is a finite sum of irreducible discrete series characters* $\delta_i^{\mathbf{H}}$ *whose generic central characters* $gcc^{\mathbf{H}}(\delta_i^{\mathbf{H}})$ *constitute one* $W_{s(e)}$*-orbit of a generic linear* $R_{s(e),1}$*-residual point* ξ *(using Theorem 7.1). We express this by writing*

$$gcc^{\mathbf{H}}(\delta^{\mathbf{H}}) = W_{s(e)}\xi \tag{124}$$

With this notation the bijection above has the property that if $\delta \in \Delta^{s(e)}(\mathcal{R}, q)$ *corresponds to* $\delta^{\mathbf{H}} \in \Delta(\mathbf{H}_e)$ *with* $gcc^{\mathbf{H}}(\delta^{\mathbf{H}}) = W_{s(e)}\xi$ *then*

$$gcc(\delta) = W_0(s(e)\exp(\xi)) \tag{125}$$

Proof. Use Theorem 2.6 and Theorem 2.8. □

Remark 8.8. *If* $\mathcal{R}$ *is of type* $R_1^{(1)}$ *then one has* $\Gamma_{s(e)} = 1$ *for all* $\in E(C^\vee)$*. In general one needs to apply Clifford theory in order to describe the sets* $\Delta(\mathbf{H}_e)$ *in terms of the results of Theorem 7.1.*

The only non-simply laced classical case which is not of type $R_1^{(1)}$ is the case $R_0 = C_n$ and $X = Q(R_0)$ (as is clear from the examples above). In this case $\mathcal{R}^{max}$ is of $C_n^{(1)}$-type with the specialization $v_{\beta^\vee} = v_{2x_n} = 1$ (as in Example 8.3). Using the notation of Example 8.5 and (7) we see that $q_0 = q(v_{2x_n}) = 1$. Hence we have $m = m_+ = m_-$, and a group $\Gamma \simeq \mathbb{Z}/2$ acting on the spectral diagram Σ as shown in Figure 5. In the application of Theorem 8.7 everything is straightforward except

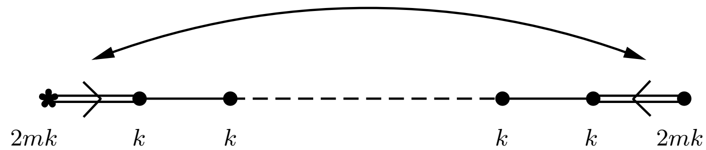

FIGURE 5. Spectral diagram for $R_0 = C_n$ and $X = Q(R_0)$

when $n = 2a$ is even and $e = e_a$ corresponds to the the middle node of Σ (the unique node of Σ with nontrivial isotropy in Γ). In this case we need to describe the set

$$\Delta(\mathbf{H}_{e_a}) = \Delta((\mathbf{H}(C_a, V_a, F_a, k_a) \otimes \mathbf{H}(C_a, V_a, F_a, k_a)) \rtimes \Gamma) \tag{126}$$

where the nontrivial element of Γ acts by the flip τ of the two tensor legs.

Theorem 8.9. *We have*

$$\Delta(\mathbf{H}_{e_a}) \simeq \Gamma \backslash (\Delta^{\mathbf{H}}(C_a, V_a, F_a, k_a) \times \Delta^{\mathbf{H}}(C_a, V_a, F_a, k_a))^\bullet \tag{127}$$

where for any set A, $(A \times A)^\bullet$ denotes the Cartesian product of A with itself with the diagonal counted twice, and where the unique nontrivial element $\gamma \in \Gamma$ acts by $\pi(\gamma)(\delta_1, \delta_2) = (\delta_2, \delta_1)$.

Proof. By Clifford theory it is clear that all irreducible discrete series representations of $\mathbf{H}_{e_a}$ are obtained by the following recipe. We start from an irreducible discrete series character $\delta = \delta_1 \otimes \delta_2$ of $\mathbf{H}(C_a, V_a, F_a, k_a) \otimes \mathbf{H}(C_a, V_a, F_a, k_a)$. Consider its inertia group for the action of Γ on such characters (by twisting). In this simple situation we see that we can choose an explicit intertwining isomorphism

$$\pi(\gamma) : \delta_1 \otimes \delta_2 \to (\delta_2 \otimes \delta_1) \circ \tau \tag{128}$$

given by $\pi(\gamma)(v \otimes w) = w \otimes v$. Hence the inertia subgroup in Γ of $\delta_1 \otimes \delta_2$ is nontrivial iff δ_1 and δ_2 are equivalent irreducible representations. If the inertia is trivial then Clifford theory tells us that the induction of $\delta_1 \otimes \delta_2$ to $\mathbf{H}_{e_a}$ is irreducible, and otherwise Clifford theory tells us that the induced representation splits up in two inequivalent irreducible parts (distinguished from each other by the sign of the trace of γ). This proves the result. □

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